Multipole as an Element of Electric Circuit and Its Equivalent Diagrams

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***Abstract* – A multipole in electric circuit theory (unlike to electric or magnetic field multipole calculation methods) is a circuit or a part of a circuit, which is connected to other circuits by a definite number of terminals (poles). An expression is given for determining the number of parameters of the equivalent diagramof a multipole depending on the number of poles. Unlike the generally known four-pole theory that can be used only if both incoming currents of the four-pole (and the same for both outgoing currents) are mutually equal, a so called generalized four-pole is considered, which does not comply with the so-called rule of regularity. In general case, the equivalent diagram of such a generalized, passive four-pole (that conforms to the reciprocity principle) should contain six passive elements unlike the "usual" regular four-pole, equivalent diagram of which needs only three resistances.**

Mathematical analogy of an electric circuit multipole and a charged system of bodies is shown. An expression is given for determining the number of parameters of the equivalent diagram depending on the number of poles. A double T-type diagram of a generalized four-pole is considered. Equivalent diagrams of an active multipole have been created by replacing the inner sources with outer sources of current or voltage.

***Keywords* – Electric circuit; equivalent diagram; generalized four-pole; multipole.**

The term "multipole" is used in various branches of physics and engineering with different meaning assigned to it. In mechanics, a set of material points, whose objects which mass density must be expressed by Dirac δ-function, is frequently called a multipole. Similarly, in electrostatics point charges are often used as well as dipoles, quadrupoles and more complex "multipoles". Using them, methods for numerical calculations of the electric field in regions of complex configuration have been created [1] – [3]. It is stated that these methods are faster and more accurate than the finite-element method and similar calculation algorithms [4].

In electric circuit theory, a part of the circuit that can be connected with other parts of the circuit by a definite (n) amount of terminals (poles) is called a multipole (an n-terminal device). In theory, most attention is paid to two-poles and four-poles. However, generally known equations of four-poles, as well as results obtained for their series, parallel and series-parallel connections apply only to four-poles satisfying the rule of regularity. It means that instead of single poles it is possible to operate with pairs of poles having the same current. If it is not so, then the entire known "four-pole theory" is unusable. Co-operation of different pairs of poles with the external circuit has also been discussed in a previously known monograph [5], paying little attention to general cases when poles cannot be integrated in pairs.

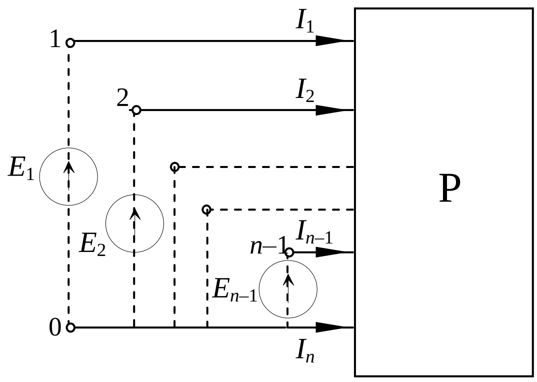

Fig. 1. Diagram for determination of multipole parameters.

Currents flow into a linear passive n-terminal device can be expressed using the superposition principle:

$$I_1 = Y_{11}\varphi_1 + Y_{12}\varphi_2 + ... + Y_{1,n-1}\varphi_{n-1};$$
$$I_2 = Y_{21}\varphi_1 + Y_{22}\varphi_2 + ... + Y_{2,n-1}\varphi_{n-1};$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$I_{n-1} = Y_{n-1,1}\varphi_1 + Y_{n-1,2}\varphi_2 + ... + Y_{n-1,n-1}\varphi_{n-1};$$

or in matrix form:

$$\boldsymbol{I} = \boldsymbol{Y}\boldsymbol{\varphi}\ .$$

If we consider a sinusoidal regime, then $I_1, I_2 ... I_n$ are complex currents; Y_{ik} are complex input and mutual admittances of the branches; but $\varphi_1, \varphi_2, ..., \varphi_{n-1}$ – complex potentials of input terminals (poles) in relation to the assumed base terminal (n). Admittances Y_{ik} can be determined experimentally (or analytically if the inner diagram of the multipole is known) as ratios I_i / E_k in a regime when a single source E_k operates in the circuit, but other terminals are connected to the base pole.

The remaining current I_n can be determined by the Kirchhoff's first law for the section surrounding the whole multipole:

$$\sum_{k=1}^{n} I_k = 0.$$

By adding and subtracting quantities $\varphi_k Y_{k1}, \varphi_k Y_{k2}, ..., \varphi_k Y_{k,n-1}$ to the expression of current I_k $(1 \le k \le n-1)$ and by grouping elements, a mathematical relationship is obtained:

$$I_k = \varphi_k \sum_{i=1}^{n-1} Y_{ki} - \sum_{i=1}^{n-1} (\varphi_k - \varphi_i) Y_{ki},$$

where an analogy with an electrostatic multipole is seen – a charged system of bodies. It is known [6] that in such a system a similar expression is derived for the charge of a body (k) if respective coefficients of the potentials are used instead of the admittances Y_{ki}. If the directions of currents are chosen to be flowing into the multipole, all the mutual admittances Y_{ki} (k≠i) are negative just as coefficients of potentials with different indexes. It is easily noticeable in Fig. 1 – if there is a single source E_k operating in the circuit, and short-circuits in place of the rest of the sources, all the other currents, except I_k, will flow in opposite to the assumed direction. The quantities $-Y_{ki}$ (k≠i) correspond to partial capacities of a system of bodies, i.e. positive quantities.

If the inner circuit of the multipole conforms to the principle of reciprocity, the matrix $\boldsymbol{Y}$ is symmetric ($Y_{ik} = Y_{ki}$). Thereby the possible number of different admittances for an n-terminal device (and the number of independent elements in the equivalent circuit) is

$$s = \frac{(n-1)(n-2)}{2} + n - 1$$

(a half of the matrix $\boldsymbol{Y}$ non-diagonal elements plus the number of diagonal elements). For a two-pole $s = 1$ (a passive two-pole is completely described by its input resistance), for an arbitrary three-pole $s = 3$, but for a four-pole where poles cannot be integrated in pairs $s = 6$ (let's call it a generalized four-pole to avoid any misunderstandings). Thereby it is seen that a 4-wire line (incl. a three-phase line with the neutral wire) needs 6 independent parameters to characterize the connected load in a general case. The number of necessary parameters can increase to $(n-1)^2$ if the multipole contains elements that don't conform to the principle of reciprocity (e.g. amplifiers).

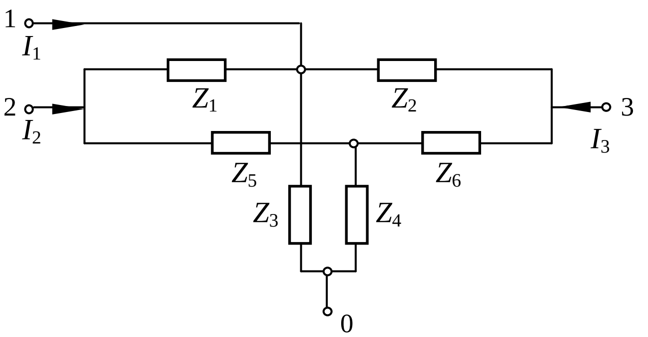

Fig. 2. Equivalent double T-type diagram of the generalized four-pole

One of the possible equivalent circuits for a generalized four-pole is the double T-diagram (fig. 2). Expressions for the circuit's input and mutual admittances Y_{ik} can be easily derived by impedances $Z_1 \ldots Z_6$ directly from the definition of Y_{ik} as ratios of respective currents and voltages:

$$Y_{11} = 1/Z_1 + 1/Z_2 + 1/Z_3;\quad Y_{12} = -1/Z_1;\quad Y_{13} = -1/Z_2;$$

$$Y_{21} = -1/Z_1;\quad Y_{22} = 1/Z_1 + (Z_4 + Z_6)/\rho;\quad Y_{23} = -Z_4/\rho;$$

$$Y_{31} = -1/Z_2;\quad Y_{32} = -Z_4/\rho;\quad Y_{33} = 1/Z_2 + (Z_4 + Z_5)/\rho,$$

where $\rho = Z_4 Z_5 + Z_4 Z_6 + Z_5 Z_6$.

If the admittances are known, impedances of the equivalent double T-diagram can be obtained, in their turn, by solving Y_{11}, Y_{12}, Y_{13}, Y_{22}, Y_{23} and Y_{33} equations in relation to Z_1, Z_2, Z_3, Z_4, Z_5 and Z_6.

It can be seen that

$$Z_1 = -1/Y_{12};$$

$$Z_2 = -1/Y_{13}.$$

Then

$$Z_3 = 1/(Y_{11} + Y_{12} + Y_{13}).$$

Further, we obtain from the Y_{23} equation:

$$Z_4 = -\rho Y_{23},$$

but after inserting this one in the expressions of Y_{22} and Y_{33}, Z_5 and Z_6 can also be expressed with the quantity ρ and the given admittances:

$$Z_5 = \rho (Y_{13} + Y_{23} + Y_{33});$$

$$Z_6 = \rho (Y_{12} + Y_{22} + Y_{23}).$$

So

$$\rho = \rho^2 [-Y_{23}(Y_{13} + Y_{23} + Y_{33}) - Y_{23}(Y_{12} + Y_{22} + Y_{23}) + \\ + (Y_{13} + Y_{23} + Y_{33})(Y_{12} + Y_{22} + Y_{23})].$$

After modifications, we obtain an expression of ρ with the given admittances:

$$\rho = \left[(Y_{12} + Y_{22})(Y_{13} + Y_{33}) - Y_{23}^2\right]^{-1}.$$

This allows calculating Z_4 and Z_5 as well as Z_6.

For an active multipole the equations of current have to be complemented with addends created by inner sources: I_{1a}, I_{2a}, ..., $I_{n-1,a}$ or I_{ka} ($1 \le k \le n-1$). The expression of current I_k can be modified as

$$I_k - I_{ka} = Y_{k1}\varphi_1 + Y_{k2}\varphi_2 + \ldots + Y_{k,n-1}\varphi_{n-1}.$$

Evidently, the equation coincides with the equation of such a passive multipole which input currents are $I_k - I_{ka}$. Since currents of the active multipole are I_k, then current sources have to be added to the passive multipole in the equivalent diagram. Currents of these sources are equal to I_{ka} or I_{1a}, I_{2a}, ..., $I_{n\text{-}1,a}$. Direction of the sources is determined by Kirchhoff's first law for nodes where they are connected to (Fig. 3).

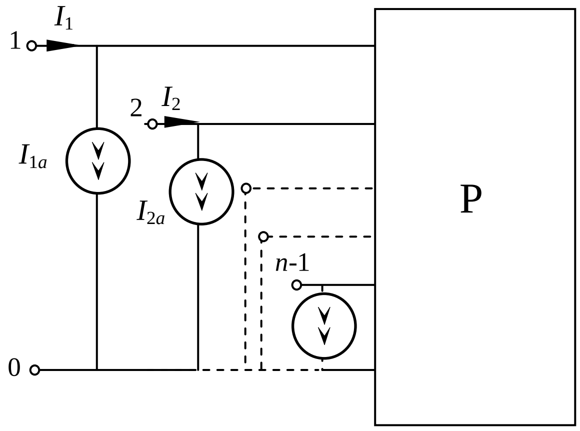

Fig. 3. Equivalent diagram of an active multipole with current sources

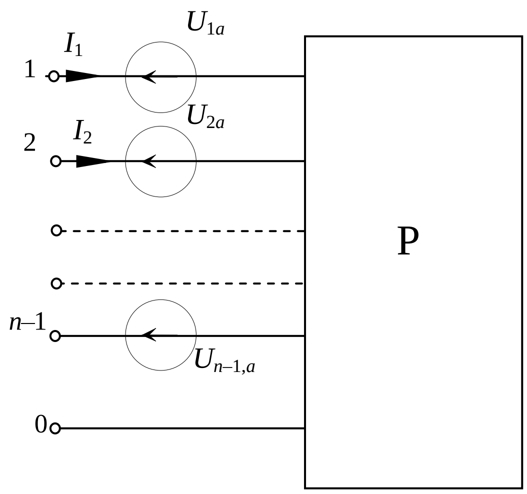

Fig. 4. Equivalent diagram of an active multipole with voltage sources

It is possible to take into account the influence of inner sources also with voltage sources. In a regime when an interruption is made on all terminals of the multipole simultaneously ($I_k = 0$, $1 \le k \le n\text{-}1$):

$$0 = Y_{k1}\varphi_{1a} + Y_{k2}\varphi_{2a} + \ldots + Y_{k,\, n-1}\varphi_{n-1,a} + I_{ka},$$

where φ_{1a}, φ_{2a} etc. is the potential on the terminals created by the inner sources. So the current created by the inner sources can be expressed as:

$$I_{ka} = -\sum_{i=1}^{n-1} \varphi_{ia} Y_{ki}.$$

Let U_{1a}, U_{2a} etc. be the voltage between the respective terminal and the terminal with zero potential when all currents are interrupted. Then $U_{1a} = \varphi_{1a}$, $U_{2a} = \varphi_{2a}$ etc. Inserting the obtained expression of I_{ka} into expression of the multipole current I_k and after modifications we obtain:

$$I_k = \sum_{i=1}^{n-1} \varphi_i Y_{ki} - \sum_{i=1}^{n-1} \varphi_{ia} Y_{ki} = \sum_{i=1}^{n-1} (\varphi_i - U_{ia}) Y_{ki}.$$

The expression coincides with the equation of such a passive multipole, which has voltages on terminals φ_i–U_{ia}. To obtain the equivalent diagram of a multipole, voltage sources have to be added to a passive multipole so that the voltage created by these sources is U_{1a}, U_{2a}, ..., $U_{n-1,a}$ as seen in Fig. 4.

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Zane Broka, Ivars Dūmiņš. Daudzpols kā elektriskās ķēdes elements un tā ekvivalentās shēmas.
Par daudzpolu var uzskatīt jebkuru elektrisko ķēdi vai tās daļu, kuru ar citām ķēdēm vai ķēdes daļām savieno noteikts savienojumu (polu) skaits. Vispārīgā gadījumā poli nav apvienojami pāros, pa kuriem plūstu vienādas strāvas (regularitātes noteikums), kā tas tiek pieņemts, piemēram, vispārzināmajā četrpolu teorijā. Darbs uzskatāms par nelielu papildinājumu visai plašajai elektrisko ķēžu teorijai.
Izmantojot superpozīcijas principu, uzrakstīti lineāra pasīva daudzpola vienādojumi un noteikts nepieciešamais ekvivalentās shēmas savstarpēji neatkarīgo elementu skaits atkarībā no polu skaita (gadījumiem, kad daudzpols pakļaujas savstarpīguma principam). Atšķirībā no regulāra četrpola, kura ekvivalentajā shēmā pietiek ar trijiem elementiem, „vispārinātam" četrpolam vispārīgā gadījumā vajadzīgi seši elementi. Aplūkota viena no iespējamām šāda četrpola ekvivalentajām shēmām – dubultā T-veida shēma un iegūtas sakarības, no kurām var noteikt tās parametrus, ja zināmas polu (zaru) ieejas un savstarpējās vadītspējas. Parādīts arī, kā aktīva daudzpola gadījumā iekšējos avotus var aizvietot ar ārējiem strāvas vai sprieguma avotiem.
Parādīta daudzpola matemātiskā analoģija ar uzlādētu ķermeņu sistēmu elektrostatikā. Daudzpolā ieplūstošās strāvas atbilst ķermeņu sistēmas lādiņiem, bet polu ieejas un savstarpējās vadītspējas – ķermeņu sistēmas potenciālu koeficientiem. No šīm vadītspējām iespējams izveidot arī lielumus, kas atbilst ķermeņu daļējām kapacitātēm.

Зане Брока, Иварс Думиньш. Многополюсник как элемент электрической цепи и его эквивалентные схемы.
Многополюсником можно считать любую электрическую цепь или ее часть, которую с другими частями соединяет определенное число соединений (полюсов). В общем случае полюсы нельзя объединить в пары полюсов, через которые протекали бы одинаковые токи (условие регулярности), как это принимается, например, в общеизвестной теории четырехполюсников. Работу можно рассматривать как небольшое дополнение к общей теории электрических цепей.
При помощи принципа наложения написаны уравнения линейного пассивного многополюсника и в зависимости от числа полюсов определено число необходимых элементов эквивалентной схемы (для случаев, когда многополюсник подчиняется принципу взаимности). В отличии от регулярного четырехполюсника, в эквивалентной схеме которого достаточно иметь 3 элемента, для «обобщенного» четырехполюсника в общем случае необходимы 6 элементов. Рассмотрена одна из возможных эквивалентных схем такого четырехполюсника – двойная Т-образная схема и получены зависимости, по которым можно рассчитать параметры элементов этой схемы, если известны собственные и взаимные проводимости входных ветвей многополюсника. Показано также, как в случае активного многополюсника внутренние источники можно заменить внешними источниками тока или напряжения.
Показана математическая аналогия многополюсника с системой заряженных тел в электростатике. При этом токи, втекающие в многополюсник, соответствуют зарядам тел, а собственные и взаимные проводимости входных ветвей – потенциальным коэффициентам системы тел. Из этих проводимостей можно образовать также величины, соответствующие частичным емкостям тел.