Scientific Journal of Riga Technical University
Computer Science. Technologies of Computer Control
2011
Volume 48

The Cox-Ingersoll-Ross Model Stationary Distribution as a Solution of the Kolmogorov Equation

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Abstract. The models of term structure of interest rates are probably the most computationally difficult part of the modern finance due to a relative complicity of application techniques. The author provides two specific term structure models and investigates the stationary probability distribution of Cox-Ingersoll-Ross model with Kolmogorov transition equation as a necessary solution for implementation of the mentioned model into MATLAB environment, in order to create simple and useful tool for simulating an adequate and accurate forecasts of interest rates dynamics.

Key words: models of term structure of interest rates, Vasicek model, Cox-Ingersoll-Ross model, Kolmogorov equation

I. INTRODUCTION

Models of term structure of interest rates are relevant and useful applications for understanding interest rate dynamics and generating future scenarios for it, for example, by testing and simulating financing strategies of ecomomy; estimating value at risk and performig hedging strategies of financial assets; pricing securities and other financial instruments as well as other tasks and decisions in finance engineering. Even for financial institutions and banks that have a wide spectrum of excellent information systems and are interested in advanced term structure modelling, this task remains quite difficult. Only some of information systems have integrated applications for advanced users that allow performing the calculations mentioned above and are considered to be additions for an extra price. Likewise, the internal development of such models for each interested means extra costs for information systems and possibly salary for specialists. One or several models that represent stochastic nature of the term structure of interest rates could be demanded, where fairly simple technics and implementation generate adequate and accurate forecasts of future interest rate scenarios without user's special knowledge of stohastic calculus and techniques of model's implementation.

In practice, there are many complications caused by a number of probable models, techniques and market aspects. In this work would be investigated an individual case – Cox-Ingersoll-Ross model (CIR) from class of affine term structure models of interest rate – where the particular attention will be dedicated to founding the stationary distribution by solving the Kolmogorov equation. This could allow creating a mathematical shape of the CIR model for implementation in MATLAB.

As a starting point on term structure of interest rates modelling a short review of term structure fundamentals and models is necessary. The risk-free pure discount interest rate (r_t often called as a interest rate for the zero-coupon bond) for time t can be calculated from the simple discount formula (1):

$$P_0 = \frac{1\$}{(1+r_t)^t}\ , \qquad (1)$$

where P_0 is the market price for contract that with certainty pays one unit of currency at its maturity date (time t from now).

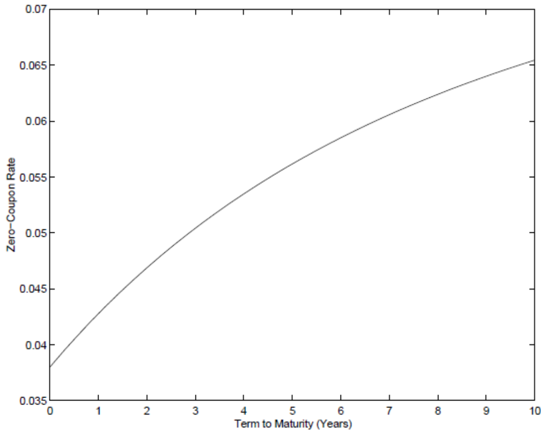

Fig. 1. The Term structure at an Instant in Time

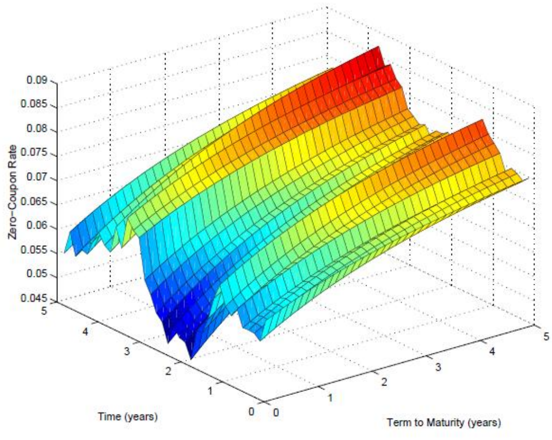

Fig. 2. Term structure dynamics over Time

Relationship between the interest rate and its term to maturity t is generally called the term structure of interest rate and can be represented with the curve r_t as a function of t (Fig. 1.). Let us assume that we know the current level of interest rates at a given instant time, but in the future given curve is liable to change, generating a steepening, flattening or inversion of the curve (Fig. 2.). Thus r_t is a random variable, it is assummed that the future dynamics of the term structure of interest rates depends on the evolution of some factor that follows a stochastic process.

An important and useful concept in the modelling of interest rates is an instantaneous interest rate. A heuristic way to consider this concept is that the interest rate demanded over an extremely short period of time (the instantaneous interest rate) does not exist in reality, but in practice, this is analogous to the overnight interest rate. The instantaneous interest rate is a concept that takes an inherently discrete-time object such as interest rate and gives it time-continuity.This permits us to use the calculus of continuous-parameter stochastic processes in modelling of the term structure of interest rates.

There are three major groups of methods, which are used for modelling of term structure of interest rates: polynomial and spline methods, stochastic factor and general equilibrium methods. Polynomial and spline methods are based on leveling of current term structure of interest rates without examining the factors that influence term structure of interest rates. The development of stochastic modelling methods began in 1977 with O.Vasicek article that had given a start for investigations in the mentioned research direction. Stochastic factor and general equilibrium methods are based on interest rate term structure modelling depending on the factors that influence it (Vasicek, Nelson-Siegel, Svensson, Longstaff-Schwartz, Cox-Ingersoll-Ross models and etc.). Frequently, researchers don't make difference between stochastic factor and general equilibrium methods with the same mathematical apparatus and difference exists only in a necessary assumption on: whether interest rates of financial instruments with different maturities are endogenous variables or whether they are exogenously raised.

II. VASICEK'S MODEL

Vasicek's model is one of the most widely-used term structure models of interest rates. According to Vasicek, the market is effective, information is equally available to all market participants and they act rationally by preferring the highest level of wealth using all available information. No transaction costs are considered. In this case, market participants have the same hopes and risk-free arbitrage is impossible, which gives the expected profit. Value of short term spot rate is the only variable (factor), which provides possible term structure. Vasicek assumes that the instantaneous interest rate follows a mean reverting process also known as an Ornstein-Uhlenbeck process (2):

$$dr(t) = k(\vartheta - r)dt + \sigma dW \ , \qquad (2)$$

where the instantaneous drift $k(\vartheta - r)$ represents a force that keeps pulling the short rate towards its long-term mean ϑ with a speed k proportional to the deviation of the process from the mean. The stochastic element σdW, which has a constant instantaneous varinace σ^2 (a variance per unit of time dt) causes the process to fluctuate around the level ϑ in a erratic, but continuous, fashon. If σ is high, this means that the value of r quickly returns to its average value and the value of r most of the time remains close to the value of ϑ. dW itself is a standart Wiener process where $dW \sim N\left(0, \sqrt{dt}\right)$, this stands that in Vasicek's model the instantaneous interest rate r is changing constantly according to the normal probability distribution. This process is a continuous time analogue to an auto-regressive process.

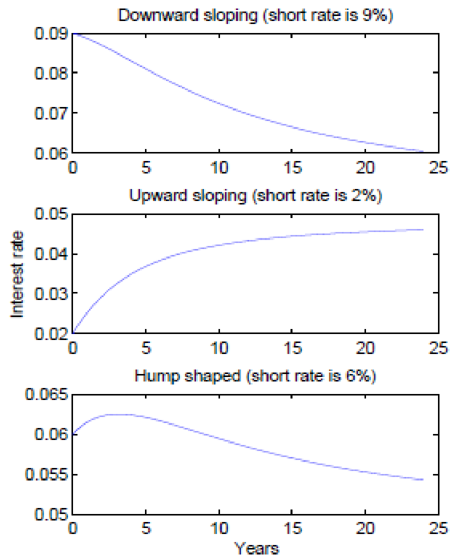

Fig. 3. Different shapes of the term structure using the Vasicek's model

A major advantage of the Vasicek's model compared to other models is that this model is able to reconstruct different shapes of the term structure, which sometimes occur in reality. Mostly, for a large part this is due to the mean reversion which is captured by the drift $k(\vartheta - r)$ in Vasicek's model (Fig. 3.), up- or downwarding shaped, humped shaped curves are computed by taking different parameter values in model).

A drawback of the Vasicek's model is that the model can produce negative interest rates. If real interest rates (corrected for inflation) have to be modeled, this will be a big problem as real interest rates can't be negative in reality. Nominal rates, on the contrary, will never be negative in practice.

Cox, Ingersoll and Ross (1985) in their own model have transformed the Vasicek's model to prevent the short rates from negative values.

III. COX-INGERSOLL-ROSS MODEL

One of the most important term structure models in the literature is the Cox-Ingersoll-Ross (1985) model. In this model, the dynamics of the short rate r, are governed by the stochastic differential equation (3):

$$dr = k(\vartheta - r)dt + \sigma\sqrt{r}dW \ , \qquad (3)$$

where dW is a standard Wiener process, and k, ϑ and σ are positive constants. Because of the drift term $k(\vartheta - r)$, the short rate process is mean reverting; the current value of the short rate process is pulled towards the long-term mean ϑ with a speed proportional to the difference from the mean. The volatility term, $\sigma\sqrt{r}$, approaches zero as r, ensuring that the short rate stays positive. Also, the volatility increases as the short rate increases. The process given in (3) is time-homogenous Markov process (for which conditional on the present state of the system, future and past are independent).

Let us start with a more general process by assuming that the interest rate follows the process (4):

$$dr = \mu(r)dt + \sigma(r)dW\ , \tag{4}$$

where $\mu(r) = k(\vartheta - r)$ and $\sigma(r) = \sigma\sqrt{r}$. The starting point is to establish the Kolmogorov (forward) transition equation to describe the evolution of probability distribution function $\varphi(r_0;t;r;T)$ of the interest rate that follows the process given by (5):

$$\begin{aligned}&\frac{1}{2}\frac{d^2}{dr^2}\left(\sigma^2(r)\varphi(r_0;t;r;T)\right)-\\&-\frac{d}{dr}\left(\mu(r)\varphi(r_0;t;r;T)\right)=\\&=\frac{d}{dt}\varphi(r_0;t;r;T)\end{aligned} \tag{5}$$

In a steady-state equilibrium the probability distribution function $\frac{d}{dt}\varphi(r_0;t;r;T)=0$ will settle down to a distribution $\varphi_\infty(r)$ which independent of the initial value of the rate and of time such that the distribution satisfies an ordinary differential equation (6):

$$\frac{1}{2}\frac{d^2}{dr^2}\left(\sigma^2(r)\varphi_\infty(r)\right)=\frac{d}{dr}\left(\mu(r)\varphi_\infty(r)\right) \tag{6}$$

Integrating both sides, this becomes (7):

$$\frac{d}{dr}\left(\frac{1}{2}\sigma^2(r)\varphi_\infty(r)\right)=\mu(r)\varphi_\infty(r)+C_1 \tag{7}$$

Our objective is to solve the differential equation in (7) to eventually obtain the density function $\varphi_\infty(r)$.

These are reasonable assumption for a process: $r > 0$ and $\varphi_\infty(0) = 0$. In this part it is necessary to make a small deviation. In some of the models the stochastic process postulated for the interest rate allows the rate to become negative. Truly, the real interest rate can be negative by pricing real bonds, but by trying to price nominal bonds, it is an improbable assumption. In the CIR model this problem is prevented by postulating an interest rate process which cannot become negative. To prove this assumption it is necessary to solve the stochastic second order differential equation to calculate average exit time of process border $\nu(r)$. If calculated time aspires to infinity, assumption of CIR model that $r > 0$ and $\varphi_\infty(0) = 0$ is true.

$$\left[\frac{\left(\sigma\sqrt{r}\right)^2}{2}\frac{d^2}{dr^2}+k(\vartheta-r)\frac{d}{dr}\right]\nu(r)=-1 \tag{8}$$

$$\begin{aligned}\nu(r) = Z_3\int\Gamma\left(\frac{2k\vartheta}{\sigma^2},\frac{2kr}{\sigma^2}\right)e^{\frac{2k\vartheta}{\sigma^2}\left(\log r-\frac{r}{\vartheta}\right)}dr+\\+Z_1\int e^{-\frac{2k\vartheta}{\sigma^2}\left(\log r-\frac{r}{\vartheta}\right)}dr+Z_2\end{aligned} \tag{9}$$

Substituting assumption $\mu(r) = k(\vartheta - r)$ and $\sigma(r) = \sigma\sqrt{r}$ in (7) and $\varphi_\infty(0) = 0$ in (10), obtains (11), where $C_1 = 0$ (10,11):

$$\frac{d}{dr}\left(\frac{1}{2}\sigma^2(r)\varphi_\infty(r)\right)=k(\vartheta-r)\varphi_\infty(r)+C_1 \tag{10}$$

$$\underbrace{\frac{d}{dr}\left(\frac{1}{2}\sigma^2(r)\varphi_\infty(0)\right)}_{0}=\underbrace{k(\vartheta-r)\varphi_\infty(0)}_{0}+C_1 \tag{11}$$

A differential equation (12) is easier to solve:

$$\frac{d}{dr}\left(\frac{1}{2}\sigma^2 r\varphi_\infty(r)\right)=k(\vartheta-r)\varphi_\infty(r) \tag{12}$$

Further mentioned equations ((13) – (24)) demonstrate steps for solving of differential equation in (12), to get as a result a mathematical shape of stationary distribution function of CIR model (25):

$$\begin{aligned}&\frac{d}{dr}\left(\frac{1}{2}\sigma^2 r\right)\varphi_\infty(r)+\\&+\frac{1}{2}\sigma^2 r\frac{d\varphi_\infty(r)}{dr}=\\&=k(\vartheta-r)\varphi_\infty(r)\end{aligned} \tag{13}$$

$$\begin{aligned}&\frac{1}{2}\sigma^2\varphi_\infty(r)+\\&+\frac{1}{2}\sigma^2 r\frac{d\varphi_\infty(r)}{dr}=\\&=k(\vartheta-r)\varphi_\infty(r)\end{aligned} \tag{14}$$

$$\begin{aligned}&\frac{1}{2}\sigma^2 r\frac{d\varphi_\infty(r)}{dr}=\\&=k(\vartheta-r)\varphi_\infty(r)-\\&-\frac{1}{2}\sigma^2\varphi_\infty(r)\end{aligned} \tag{15}$$

$$\begin{aligned}&\frac{1}{2}\sigma^2 r\frac{d\varphi_\infty(r)}{dr}=\\&=\varphi_\infty(r)\left(k(\vartheta-r)-\frac{1}{2}\sigma^2\right)\end{aligned} \tag{16}$$

$$\begin{aligned}&\frac{1}{2}\sigma^2 r\frac{d\varphi_\infty(r)}{dr}=\\&=\varphi_\infty(r)\left(\frac{2k(\vartheta-r)-\sigma^2}{2}\right)\end{aligned} \tag{17}$$

$$\frac{d\varphi_\infty(r)}{dr}=\varphi_\infty(r)\left(\frac{2k(\vartheta-r)-\sigma^2}{\sigma^2 r}\right) \tag{18}$$

$$\frac{d\varphi_\infty(r)}{\varphi_\infty(r)}=\frac{\left(2k(\vartheta-r)-\sigma^2\right)dr}{\sigma^2 r} \tag{19}$$

$$\int\frac{d\varphi_\infty(r)}{\varphi_\infty(r)}=\int\frac{\left(2k(\vartheta-r)-\sigma^2\right)dr}{\sigma^2 r} \tag{20}$$

$$\int\frac{d\varphi_\infty(r)}{\varphi_\infty(r)}=\int\left(\begin{aligned}&\frac{\left(2k\vartheta-\sigma^2\right)dr}{\sigma^2 r}-\\&-\frac{(2kr)dr}{\sigma^2 r}\end{aligned}\right) \tag{21}$$

$$\begin{aligned}&\int\frac{d\varphi_\infty(r)}{\varphi_\infty(r)}=\int\left(\frac{2k\vartheta}{\sigma^2}-1\right)\frac{dr}{r}-\\&-\int\left(\frac{2kr}{\sigma^2}\right)\frac{dr}{r}\end{aligned} \tag{22}$$

$$\begin{aligned}&\ln\varphi_\infty(r)=\left(\frac{2k\vartheta}{\sigma^2}-1\right)\ln r-\\&-\frac{2kr}{\sigma^2}r+C_2\end{aligned} \tag{23}$$

$$\ln\varphi_\infty(r)=r^{\left(\frac{2k\vartheta}{\sigma^2}-1\right)}-\frac{2k}{\sigma^2}r+\ln C_2 \tag{24}$$

In (25) is given the general shape of the unconditional density function of a variable r that follows the process given in (4):

$$\varphi_\infty(r)=C_2 r^{\left(\frac{2k\vartheta}{\sigma^2}-1\right)}e^{-\frac{2k}{\sigma^2}r} \tag{25}$$

The constants of integration C_1 and C_2 are determined to guarantee that:

If $r\geq 0$ then $\varphi_\infty(r)\geq 0$ and $\int_{-\infty}^{+\infty}\varphi_\infty(r)dr=1$ (26)

If $r>0$ then $\varphi_\infty(r)>0$ and $\int_{0}^{+\infty}\varphi_\infty(r)dr=1$ (27)

The steady-state distribution for the interest rate following the process given in (4) is:

$$\varphi_\infty(r)=\frac{r^{\left(\frac{2k\vartheta}{\sigma^2}-1\right)}e^{-\frac{2k}{\sigma^2}r}}{\dfrac{\sigma^{2\frac{2k\vartheta}{\sigma^2}}}{2k}\Gamma\left(\dfrac{2k\vartheta}{\sigma^2}\right)} \tag{28}$$

where C_2 was calculated by following:

$$C_2=\frac{1}{\underbrace{\int_0^{+\infty}r^{\left(\frac{2k\vartheta}{\sigma^2}-1\right)}e^{-\frac{2k}{\sigma^2}r}dr}_{Gamma function}} \tag{29}$$

Gamma function general form is given in (30):

$$\Gamma(x) = \int_{0}^{+\infty} e^{-z} z^{x-1} dz \,, \tag{30}$$

where $z = \frac{2k}{\sigma^2} r$, $r = \frac{\sigma^2}{2k} z$ and $x = \frac{2k\vartheta}{\sigma^2}$. By ssubstituting assumptions in equation (30) follows:

$$\int_{0}^{+\infty} r^{\left(\frac{2k\vartheta}{\sigma^2}-1\right)} e^{-\frac{2k}{\sigma^2}r} dr =$$

$$= \int_{0}^{+\infty} r^{\left(\frac{2k\vartheta}{\sigma^2}-1\right)} e^{-\frac{2k}{\sigma^2}r} \left(\frac{\sigma^2}{2k}\right) d\left(\frac{2k}{\sigma^2} r\right) =$$

$$= \frac{\sigma^2}{2k} \int_{0}^{+\infty} \left(\frac{\sigma^2}{2k} z\right)^{\left(\frac{2k\vartheta}{\sigma^2}-1\right)} e^{-z} dz =$$

$$= \frac{\sigma^2}{2k} \int_{0}^{+\infty} \left(\frac{\sigma^2}{2k}\right)^{\left(\frac{2k\vartheta}{\sigma^2}-1\right)} z^{\left(\frac{2k\vartheta}{\sigma^2}-1\right)} e^{-z} dz =$$

$$= \frac{\sigma^2}{2k}^{\frac{2k\vartheta}{\sigma^2}} \int_{0}^{+\infty} z^{x-1} e^{-z} dz \tag{31}$$

IV. SUMMARY

Models of term structure of interest rates are relevant and useful applications for understanding interest rate dynamics and are probably the most computationally difficult part of the modern finance due to a relative complicity of application techniques and other aspects. Thereof, numbers of information systems have integrated applications that allow performing stochastic calculus and other calculations needed for modelling the term structure of interest rates, and these software programmes are usually very sophisticated for users and expensive at the same time.

The author provides a short review of term structure fundamentals and major groups of term structure modelling methods such as polynomial and spline methods, stochastic factor and general equilibrium methods. Continuing the author gives an overview of two specific term structure models: the Vasicek's model, as an example of one-factor stochastic models, and the Cox-Ingersoll-Ross model, as an example of general equilibrium models (also known as *affine* models). Considering the fact, that there is no tremendous difference between the methods of term structure modelling mentioned above, the Cox-Ingersoll-Ross model was chosen for integration in MATLAB (by the reason of using the same origin basis as in Vasicek's model and some differencies in a assumption on variables. The Cox-Ingersoll-Ross model, in comparison to Vasicek's model, is preventing the negative interest rate forecasts, and hereof, more truly reflects the dynamics of interest rates in real financial markets. In order to create simple and useful tool for simulating an adequate and accurate forecasts of interest rates dynamics the special attention has been dedicated to founding the stationary probability distribution of the Cox-Ingersoll-Ross model with Kolmogorov transition equation as a necessary solution for implementation of mentioned model into MATLAB environment.

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Oļesja Zamovska, Koksa-Ingersolla-Rossa modeļa stacionārs sadalījums kā Kolmogorova vienādojuma risinājums
Procentu likmju termiņstruktūras modeļi ir nozīmīgas un noderīgas programmaplikācijas, lai saprastu procentu likmju dinamiku, un, iespējams, aprēķinu ziņā viena no sarežģītakajām jomām mūsdienu finansēs pietiekami sarežģītas piemērošanas tehnikas un citu aspektu dēļ. Tā rezultātā, tikai dažām informācijas sistēmām ir integrētās programmaplikācijas, kas ļauj veikt stohastiskos un citus aprēķinus, kas ir nepieciešami procentu likmju termiņstruktūras modelēšanai, un šīs programmas parasti ir ļoti sarežģītas lietotājiem un arī dārgas. Autore sniedz īsu ieskatu procentu likmju termiņstruktūras pamatos un modelēšanas metožu pamatgrupās: polinomu un splainu metodes, stohastiskās faktoru un vispārējā līdzsvara metodes. Darba gaitā autore apskata divus specifiskus procentu likmju termiņstruktūras modeļus: Vasičeka modelis, kā vienfaktora stohastisko modeļu piemērs, un Koksa-Ingersolla-Rossa modelis, kā vispārējā līdzsvara modeļa piemērs. Ņemot vērā faktu, ka starp iepriekšminētajām procentu likmju termiņstruktūras modelēšanas metodēm nav lielu atšķirību to izcelsmes dēļ, bet atšķirības pieņēmumos par mainīgajiem lielumiem nav būtiskas, integrācijai MATLAB vidē tika izvēlēts Koksa-Ingersolla-Rossa modelis. Koksa-Ingersolla-Rossa modelis salīdzinājumā ar Vasičeka modeli novērš negatīvu procentu likmju rašanos prognozēs, un tāpēc patiesāk atspoguļo procentu likmju reālo uzvedību un dinamiku finanšu tirgos. Īpaša uzmanība tika veltīta, lai noteiktu stacionāro varbūtību sadalījumu Koksa-Ingersolla-Rossa modelim, risinot Kolmogorova vienādojumu un

šādi nosakot matemātisko formu modeļa izstrādei MATLAB vidē, lai izveidotu vienkāršu un noderīgu rīku, kas palīdzēs modelēt adekvātas un precīzākas procentu likmju dinamikas prognozes.

Олеся Замовска, Стационарное распределение модели Кокс-Ингерсолл-Росс как решение уравнения Колмогорова

Модели временной структуры процентных ставок являются важными и полезными аппликациями для понимания динамики процентных ставок и, возможно, с точки зрения вычислений, одной из самых сложных областей современных финансов, благодаря достаточно сложным применяемым техникам и другим аспектам. Вследствие этого, лишь небольшое число информационных систем имеют встроенные программные приложения, которые позволяют выполнять стохастические вычисления и другие расчеты, необходимые при моделировании временной структуры процентных ставок, и такое программное обеспечение, как правило, достаточно сложное для пользователей и в то же время дорогое. Автор делает краткий обзор основ временной структуры процентных ставок и основных методов их моделирования: методы полиномов и сплайнов, стохастические факторные методы и методы общего равновесия. Далее, автор рассматривает две конкретные модели временной структуры процентных ставок: модель Vasicek, как пример однофакторной стохастической модели, и модель Cox-Ingersoll-Ross, как пример модели общего равновесия (также известной как *affine* модель). Учитывая тот факт, что существует лишь незначительная разница между выше упомянутыми методами моделирования временной структуры процентных ставок по причине их происхождения, а также несущественные отличия в переменных, модель Cox-Ingersoll-Ross была выбрана для интеграции в среде MATLAB. Модель Cox-Ingersoll-Ross, по сравнению с моделью Vasicek, не допускает появление негативных процентных ставок при прогнозировании, что наилучшим образом отображает динамику процентных ставок на финансовых рынках. Особое внимание было уделено формулировке стационарного распределения вероятностей для данной модели посредством решения уравнения Колмогорова для получения математической формы Cox-Ingersoll-Ross модели для последущей разработки в среде MATLAB и создания простого и полезного инструмента для моделирования адекватных и аккуратных прогнозов динамики процентных ставок