Ginzburg-Landau Model for Stability Analysis of Fluid Flows

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Abstract **- A general scheme for the solution of stability problems for two-dimensional flows (the Navier-Stokes equations and shallow water equations) by means of a weakly nonlinear theory is analyzed in the paper. Equations of the first, second and the third order are presented using a perturbation expansion of the stream function of the flow and the method of multiple scales. It is shown that the amplitude evolution equation for the amplitude of the most unstable mode is the complex Ginzburg-Landau equation. The equation is derived using solvability condition at the third order. Possible applications of the Ginzburg-Landau model are discussed in the paper**.

Keywords **- Ginzburg-Landau equation, weakly nonlinear instability**

I. Introduction

The first step in the stability analysis of a base flow in fluid mechanics is usually the linear stability analysis. The general scheme of the method is well-known and presented in many excellent textbooks (see, for example, [1]-[4]). Here we briefly describe the main steps in the procedure. First, a steady base flow $\vec{U}(\vec{x})$ is selected. Usually $\vec{U}(\vec{x})$ represents a relatively simple velocity profile like plane Poiseuille flow in a channel or Taylor-Couette flow between two rotating cylinders [2]-[3]. Second, small unsteady perturbations $\vec{u}'(\vec{x},t)$ are imposed on the base flow so that the total velocity vector has the form $\vec{u}(\vec{x},t)=\vec{U}(\vec{x})+\vec{u}'(\vec{x},t)$. Third, the vector $\vec{u}(\vec{x},t)$ is substituted into equations of motion (for example, the Navier-Stokes equations of viscous fluid flow). Next, the equations of motion are linearized in the neighbourhood of the base flow, that is, all nonlinear terms with respect to $\vec{u}'(\vec{x},t)$ are neglected. Finally, solution of the linearized equations is sought by the method of normal modes in the form

$$\vec{u}'(\vec{x},t)=\vec{v}(\vec{x})\exp(-\lambda t), \tag{1}$$

where $\lambda=\lambda_r+i\lambda_i$ is a complex constant. Further simplifying assumptions are usually made at this stage. For example, perturbations are assumed to be periodic with respect to one or two spatial variables. The resulting ordinary differential equation for a system of ordinary differential equations together with zero boundary conditions forms an eigenvalue problem. A nontrivial solution of the eigenvalue problem exists only for some values of λ. The base flow $\vec{U}(\vec{x})$ is said to be linearly stable if all $\lambda_r>0$, and linearly unstable if at least one $\lambda_r<0$. A classical example of a linear stability problem is the Orr-Sommerfeld equation for the analysis of stability of a plane Poiseuille flow [2], [3]. In many cases a linearized problem contains a parameter γ (such as the Reynolds number R for two-dimensional viscous flows or the bed friction number S for shallow water flows). One of the objectives of linear stability analysis is to find the values of the parameter γ for which the base flow $\vec{U}(\vec{x})$ is linearly stable.

Linear stability analysis can be used in order to investigate when a particular base flow becomes unstable. Linear stability calculations provide the critical value of the parameter γ and the form of the most unstable mode. However, linear stability theory cannot predict the behaviour of the most unstable mode when the parameter γ is above the threshold.

II. Weakly Nonlinear Analysis

Weakly nonlinear theories can be used in order to analyze the development of the most unstable mode in the region where the growth rate of the most unstable perturbation is positive. However, the growth rate cannot be too large since in this case the perturbation will grow quickly, and nonlinear terms become important after a short time. Thus, weakly nonlinear theories are usually constructed in the neighbourhood of a critical point (see Fig. 1 and Fig. 2).

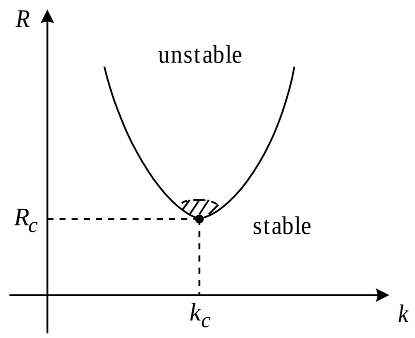

Fig. 1. A typical marginal stability curve for two-dimensional viscous flow. Here *R* is the Reynolds number and *k* is the wave number of the perturbation

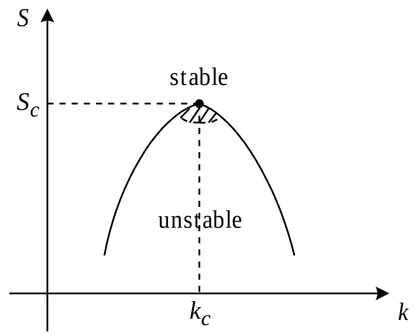

Fig. 2. A typical marginal stability curve for shallow water flow. Here *S* is the bed friction number and *k* is the wave number of the perturbation

Solid curves in Fig. 1 and Fig. 2 represent marginal stability curves (where $\lambda_r = 0$). The regions of linear stability and instability are indicated on the graphs.

We consider a small neighbourhood of the critical point (k_c, γ_c), where the growth rate $\lambda_r > 0$ is quite small (here $\gamma_c = R_c$ in Fig. 1 and $\gamma_c = S_c$ in Fig. 2). It is shown in [5]-[8] that an amplitude evolution equation in this case is the complex Ginzburg-Landau (GL) equation of the form

$$\frac{\partial A}{\partial \tau} = \sigma A + \delta \frac{\partial^2 A}{\partial \xi^2} - \mu A |A|, \qquad (2)$$

where $\sigma = \sigma_r + i\sigma_i$, $\delta = \delta_r + i\delta_i$ and $\mu = \mu_r + i\mu_i$ are complex coefficients which can be computed using linearized characteristics of the flow.

The constant μ_r is known as the Landau constant in the literature. If $\mu_r > 0$ then finite saturation of the amplitude is possible and (2) can be useful in analyzing the development of instability. However, for plane Poiseuille flow $\mu_r < 0$ (see [5]) so that (2) is not useful at all since higher-order terms become important as well.

There are many examples in fluid mechanics where the constant μ_r has the "right sign", that is, where $\mu_r > 0$. Examples include rotating convective flows [9], [10] and shallow water flows [6]-[8].

III. Derivation of the Ginzburg-Landau Equation

In this section we illustrate the basic steps of the derivation of (2) under the assumption that equations of motion can be reduced to one scalar equation with respect to the stream function of the flow (examples include two-dimensional Navier-Stokes equations and shallow water equations under the rigid-lid assumption).

Suppose that equation of motion is reduced to the form

$$N\psi = 0, \qquad (3)$$

where N is a nonlinear operator.

Consider a perturbation expansion of the form

$$\psi(x,y,t) = \psi_0(y) + \varepsilon\psi_1(x,y,t) + \varepsilon^2\psi_2(x,y,t) + \varepsilon^3\psi_3(x,y,t) + \ldots \qquad (4)$$

where $\psi_0(y)$ is the stream function of the base flow and the role of the parameter ε will be clarified later.

Consider a base flow $U(y)$. Imposing small perturbations on the flow and linearizing equations of motion in the neighbourhood of the base flow, we obtain the linearized equation for the function ψ_1:

$$L\psi_1 = 0. \qquad (5)$$

Following the method of normal modes we seek the solution to (5) in the form

$$\psi_1(x,y,t) = \varphi_1(y)\exp[ik(x-ct)] + c.c., \qquad (6)$$

where $\varphi_1(y)$ is the amplitude of the normal perturbation, c is the phase speed of the perturbation, and complex conjugate terms are denoted by $c.c.$

Substituting (6) into (5) and using zero boundary conditions for the function $\varphi_1(y)$ we obtain the eigenvalue problem of the form

$$L_1\varphi_1 = 0. \qquad (7)$$

Numerical solution of (7) gives the critical values of the parameters $k = k_c, \gamma = \gamma_c$ and $c = c_c$. The marginally stable mode (in accordance with the linear theory) is given by (6) where $k = k_c, \gamma = \gamma_c$ and $c = c_c$ and $\varphi_1(y)$ is the eigenfunction of (7). Since eigenfunctions cannot be uniquely determined, any function of the form $C\varphi_1(y)$, where C is an arbitrary constant, also is an eigenfunction of (7). Hence, any function of the form

$$\psi_1(x,y,t) = C\varphi_1(y)\exp[ik(x-ct)] + c.c. \qquad (8)$$

represents a marginally stable mode. Note that constant C cannot be determined by means of the linear stability theory.

Assume that the parameter γ is slightly different from the critical value so that the base flow is linearly unstable but the growth rate is quite small (for example, $R = R_c(1+\varepsilon^2)$ in Fig. 1 or $S = S_c(1-\varepsilon^2)$ in Fig. 2). In other words, we select the value of γ in a small neighbourhood of the critical point in the unstable region shown in Fig. 1 and Fig. 2. Note that the dependence of γ on ε is determined by the form of the equation of motion. Our goal is to introduce a slowly varying amplitude function A, which replaces C in (8), and derive an amplitude evolution equation for A.

Introducing "slow" time $\tau = \varepsilon^2 t$ and spatial coordinate $\xi = \varepsilon(x - c_g t)$, where c_g is the group velocity, substituting (4) into (3) and collecting coefficients containing the same powers of ε we obtain the sequence of linear problems

$$L\psi_1 = 0, \qquad (9)$$

$$L\psi_2 = f_1, \qquad (10)$$

$$L\psi_3 = f_2, \qquad (11)$$

and so on.

Note that the derivatives with respect to x and t in this case are replaced by

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \varepsilon c_g \frac{\partial}{\partial \xi} + \varepsilon^2 \frac{\partial}{\partial \tau},$$
$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \xi}. \quad (12)$$

Replacing constant *C* in (8) by a new amplitude function $A(\xi,\tau)$ we obtain

$$\psi_1(x,y,t) = A(\xi,\tau)\varphi_1(y) exp[ik(x-ct)] + c.c.. \quad (13)$$

The form of the solution for ψ_2 is determined by the function f_1 in (10). Similarly, the function f_2 in (11) determines the form of ψ_3 and so on. Note that the operator L on the left-hand side of each equation in (9)-(11) is the same. If $k = k_c$ and $\gamma = \gamma_c$ then homogeneous equations $L\psi_2 = 0$, $L\psi_3 = 0$ have non-trivial solutions. It follows from the Fredholm's alternative [11] that equations $L\psi_2 = f_1$, $L\psi_3 = f_2$ have solutions if and only if the functions f_1 and f_2 are orthogonal to all eigenfunctions of the corresponding adjoint problem.

The adjoint equation to (7) is

$$L_1^a \varphi_1^a = 0, \quad (14)$$

where L_1^a is the adjoint operator and φ_1^a is the corresponding adjoint eigenfunction (computational details can be found, for example, in [6]).

Analysis of the function f_1 in (10) shows that it has the form

$$f_1(x,y,t,\xi,\tau) = AA^* f_1^{(0)}(y)e^{ikx} + A_\xi f_1^{(1)}(y)e^{ik(x-ct)} + A|A| f_1^{(2)}(y)e^{2ik(x-ct)} + c.c., \quad (15)$$

where $f_1^{(0)}(y)$, $f_1^{(1)}(y)$ and $f_1^{(2)}(y)$ are known functions of y. Using (10) and (15) we conclude that ψ_2 should be sought in the form

$$\psi_2(x,y,t,\xi,\tau) = AA^* \varphi_2^{(0)}(y)e^{ikx} + A_\xi \varphi_2^{(1)}(y)e^{ik(x-ct)} + A|A| \varphi_2^{(2)}(y)e^{2ik(x-ct)} + c.c., \quad (16)$$

where $\varphi_2^{(0)}(y)$, $\varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$ are unknown functions of y. Substituting (16) into (10) and using (15) we obtain the following three boundary value problems

$$M_0 \varphi_2^{(0)} = f_1^{(0)}, \quad (17)$$

$$M_1 \varphi_2^{(1)} = f_1^{(1)}, \quad (18)$$

$$M_2 \varphi_2^{(2)} = f_1^{(2)}, \quad (19)$$

where M_0, M_1 and M_2 are linear operators. Furthermore, $f_1^{(0)}$, $f_1^{(1)}$ and $f_1^{(2)}$ are known functions of y.

Problems (17)-(19) are solved numerically. Note that equation (18) is resonantly forced since the corresponding homogeneous equation at $k = k_c$, $c = c_c$ and $\gamma = \gamma_c$ has a nontrivial solution. Thus, a singular value decomposition method [12] should be used in order to solve (18).

Solvability condition at the second order in ε can be written in the form

$$\left\langle f_1^{(1)}, \varphi_1^a \right\rangle = 0, \quad (20)$$

where $\langle \bullet,\bullet \rangle$ is a suitably defined dot product. Using (20), we obtain the group velocity c_g.

Solvability condition at the third order in ε gives equation (2) (at least this statement is true and has been verified by direct calculations for two-dimensional Navier-Stokes equation and shallow water equations under the rigid-lid assumption). The coefficients of (2) are explicitly computed as integrals containing characteristics of the linearized problem (the details can be found in [5]-[8]).

IV. Use of the Ginzburg-Landau Equation

Ginzburg-Landau equation is often used to model spatio-temporal dynamics of complex flows. The reason is that (2) exhibits a rich variety of solutions depending on the values of the coefficients σ, δ and μ. In addition, it contains the terms representing linear growth, diffusion and nonlinearity. In many cases the Ginzburg-Landau equation is used as a phenomenological model, that is, it is assumed but not derived from the equations of motion. Experimental data are often used in such cases in order to estimate the coefficients of the equation.

In other cases the Ginzburg-Landau equation can be derived from the equations of motion (examples are given in [5]-[8]). The coefficients of the equation are calculated in a closed form as integrals containing characteristics of the linearized problems.

Ginzburg-Landau equation and its properties are extensively studied in the literature (see, for example, [13] and [14]). Numerical analysis of the Ginzburg-Landau equation is simpler than numerical solution of the equations of motion. In addition, stability of some simple (for example, periodic) solutions of the Ginzburg-Landau equation allow researchers to simplify the analysis of spatio-temporal dynamics of complex flows in fluid mechanics.

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Irina Eglīte, Andrejs Koliškins. Ginzburga-Landau modeļa izmantošana šķidruma plūsmu stabilitātes analīzei
Rakstā analizēta vispārējā shēma divdimensiju plūsmu stabilitātes uzdevumu risināšanai, izmantojot vāji nelineāru stabilitātes teoriju. Rakstā kā piemērs izmantoti Navje-Stoksa vienādojumi un sekla ūdens vienādojumi. Ginzburga-Landau vienādojuma iegūšanai izmantota vairāku mērogu metode. Plūsmas funkcija tiek izvirzīta rindā pēc mazā parametra. Uzdevuma kritiskie parametri t.i., viļņa skaitlis, fāzu ātrums vai parametrs, kas raksturo plūsmu (Reinoldsa skaitlis Navje-Stoksa vienādojumiem vai parametrs, kas raksturo berzi sekla ūdens vienādojumiem) tiek iegūti no lineārās stabilitātes uzdevuma atrisinājuma. Vienādojumu risināšanai otrajā un trešajā tuvinājumā tiek izmantota Fredholma alternatīva. Atrisinājuma nosacījums otrajā tuvinājumā ļauj noteikt perturbācijas grupas ātrumu. Ir parādīts, ka visvairāk nestabilu režīmu amplitūdai (saskaņā ar linearitātes teoriju) amplitūdas evolūcijas vienādojums ir kompleksais Ginzburga-Landau vienādojums. Vienādojumu iegūst, izmantojot trešā tuvinājuma atrisināšanas nosacījumu. Vienādojuma koeficienti tiek iegūti tiešā veida (precīzāk, kā integrāļi, kas satur lineāra uzdevuma īpašfunkcijas, saistītā uzdevuma īpašfunkcijas un atrisinājumus triju parastu diferenciālvienādojumu robežuzdevumiem, kurus iegūst atrisinot uzdevumu otrajā tuvinājumā. Ginzburga-Landau vienādojumus bieži izmanto, lai modelētu sarežģītu plūsmu dinamiku telpā un laikā. Rakstā ir apskatītas iespējamie Ginzburgam-Landau modeļa pielietošanas iespējas.

Ирина Эглите, Андрей Колышкин. Модель Гинзбурга-Ландау для анализа устойчивости течений жидкости
В статье анализируется общая схема решения задач устойчивости для двумерных течений жидкости с помощью методов слабо нелинейной теории устойчивости. В качестве примеров используются уравнения Навье-Стокса и уравнения мелкой воды. Для вывода уравнения Гинзбурга-Ландау используется метод многих масштабов. Функция тока раскладывается в ряд по малому параметру, который характеризует степень надкритичности. Критические параметры задачи, т.е. волновое число, фазовая скорость возмущения и параметр, характеризующий течение (число Рейнольдса для уравнений Навье-Стокса или параметр, характеризующий трение для уравнений мелкой воды), определяются из решения задачи линейной устойчивости. Для решения уравнений во втором и третьем приближении используется альтернатива Фредгольма. Условие разрешимости во втором приближении позволяет определить групповую скорость возмущений. Показано, что эволюционное уравнение для амплитуды наименее устойчивой моды (согласно линейной теории) является комплексным уравнением Гинзбурга-Ландау. Уравнение получено с использованием условия разрешимости для третьего приближения. Коэффициенты уравнения определяются в явном виде (более точно, в виде интегралов, содержащих собственные функции линейной задачи, собственные функции сопряженной задачи и решения трех краевых задач для обыкновенных дифференциальных уравнений, полученных при решении задачи во втором приближении). Уравнение Гинзбурга-Ландау часто используется для моделирования пространственно-временной динамики сложных течений. В статье рассматриваются возможные области применения модели Гинзбурга-Ландау.