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On Equilibrium of an Adaptive Single Component Market

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Abstract **- A mathematical model of an adaptive Samuel-Marshall type single component market described by quasi-linear functional differential equations with dependent on phase coordinates and frequently switched an ergodic Markov process is presented. The proposed method is based on an averaging procedure with respect to time along the critical solutions of the generative average linear equation and with respect to the invariant measure of the Markov process. It is proved that exponential stability of the resulting deterministic equation is sufficient for exponential $p-$stability of the initial random system for all positive numbers p and for sufficiently fast switching.**

Keywords **– exponential stability, Markov process, Samuel-Marshall type single component market, projective operator**

I. Introduction

The most part of the real dynamical systems are under control of the influence of the permanent random perturbations. This circumstance burdens a lot of the possibilities of their models' quantitative analysis. Investigating asymptotic of the dynamical systems we should face the special difficulties in the given situation. Very often the describing models of the dynamical systems are complicated mathematical objects and therefore they are researched like asymptotic methods of the theory of the dynamical systems. Mainly methods such as Lyapunov method, limit theorems of the probability theory, asymptotic methods of the nonlinear oscillations theory and others are used. The basic model of the description of the system subordinated to random perturbations is random process, often – Markov process. The above-mentioned class of processes is introduced by Kolmogorov to describe dynamical systems which are under the influence of independent random perturbations occurring at different time moments. He also offered an analytical technique for the research of Markov processes – differential equations for the transition probabilities of the process.

The idea of getting of the Markov process from the dynamical system which is under control of the random influence was carried out by N. Bogolyubov and N. Krylov and further it was developed by I. Gihman.

Let us consider the $n-$dimensional functional differential equation in a quasi-linear form with a small parameter $\varepsilon \in [0,1]$

$$\frac{du^{\varepsilon}(t)}{dt} = \int_{-h}^{0} \{ dG(\theta, y(t/\varepsilon)) \} u^{\varepsilon}(t+\theta) + \varepsilon F(t, u_t^{\varepsilon}, y(t), \varepsilon), \qquad (1)$$

where $\{y(t), t \geq 0\}$ is a homogenous ergodic Feller type Markov process [1] on the probability space $(\Omega, \mathrm{F}, \mathbf{P})$ with values in the compact phase space $\mathbf{Y}$, with infinitesimal operator Q, transition probability $P(t, y, dz)$ and unique invariant measure $\mu(dy)$ satisfying the condition of exponential ergodicity, that is, there exist positive constants M and δ such that $\|P(t, y, .) - \mu\| \leq M \exp\{-\delta t\}$ for any $t \geq 0$; ε is a small positive parameter; u_t^{ε} is a part of solution defined by the equality $u_t^{\varepsilon} = \{u^{\varepsilon}(t+\theta), \theta \in [-h, 0]\}$ with some positive number h; $G(\theta, y)$ is a matrix consisting of bounded variation on θ functions; the perturbing term $F(t\varphi, y, \varepsilon)$ is a continuous mapping of the product space $\mathbf{R}_+ \times \mathbf{C}_n([-h, 0]) \times \mathbf{Y} \times [0, 1)$ to the space $\mathbf{R}^n$, satisfying $F(t, 0, y, \varepsilon) \equiv 0$ and the Lipschitz condition on the second argument for any $y \in \mathbf{Y}$, $\varepsilon \in [0, 1)$, $t \in \mathbf{R}_+$. Under these conditions the random equation (1) with the initial problem $u^{\varepsilon}(s+\theta) = \varphi(\theta)$, $-h \leq \theta \leq 0$ has [2] a unique solution $u^{\varepsilon} = \{u^{\varepsilon}(t), t \geq 0\}$ for any continuous function φ. This solution is a continuous stochastic process with probability one.

Averaging the linear part of the equation (1) according to the invariant measure of Markov process

$$\overline{G}(\theta) = \int_{\mathbf{Y}} G(\theta, y) \mu(dy),$$

one can define a generative equation for the equation (1):

$$\frac{d\overline{x}(t)}{dt} = \int_{-h}^{0} \{ d\overline{G}(\theta) \} \overline{x}(t+\theta). \qquad (2)$$

It is well known [2] that equation (2) defines in the space $\mathbf{C}_n$ a strong continuous semigroup $T(t)$ with infinitesimal operator given for sufficiently smooth function φ by

$$(\mathbf{A}\varphi)(\theta) = \begin{cases} \dfrac{d\varphi(\theta)}{d\theta}, & -h \leq \theta < 0, \\ g(\varphi), & \theta = 0. \end{cases}$$

The spectrum $\sigma(\mathbf{A})$ of this operator is given by $\sigma(\mathbf{A}) = \{z : \det\{U(z)\} = 0\}$ where

$$U(z) = Iz - \int_{-h}^{0} e^{z\theta} d\overline{G}(\theta) .$$

If $\sigma(\mathbf{A}) \cap \{z : \operatorname{Re} z > 0\} = \varnothing$ and $\sigma_0 = \sigma(\mathbf{A}) \cap \{z : \operatorname{Re} z = 0\} \neq \varnothing$ the generative equation is on the border of stability and some preliminary preparation is necessary in order to obtain the resulting averaged equation.

We will refer to the spectral subspace of the operator $\mathbf{A}$ corresponding to σ_0 as the critical subspace and to the solutions of (2) lying in the critical subspace as the critical solutions. One has to note that the selection of the first linear term in the right hand part of equation (1) can be done somewhat arbitrarily. One can add any arbitrary linear continuous mapping $\varepsilon g_1(\varphi)$ to the linear part to (1) and subtract it from the second term. Using this arbitrariness and taking into consideration that the set σ_0 consists of the finite number of points [2] $\sigma_0 = \{z_j,\ j = 1, 2, ..., m\}$, it may be assumed that the selection of the terms in the right hand part of (1) has been done in such a manner that $(\det U(z_j))' \neq 0, \quad j = 1, 2, ..., m$.

One can apply the projective operator P_0 not only on any continuous vector-function $\psi(\theta)$, but also on any vector- or matrix-valued measurable function. Primarily one needs to rewrite equation (1) in the operator form [2, 8]

$$\frac{du_t^\varepsilon}{dt} = \mathbf{A}u_t^\varepsilon + \varepsilon \mathbf{I}F(t, u_t^\varepsilon, y(t), \varepsilon) \tag{3}$$

where the matrix-valued function $\{\mathbf{I}(\theta), -h \le \theta \le 0\}$ is defined by the equality

$$\mathbf{I}(\theta) = \begin{cases} 0, & \text{if } -h \le \theta < 0, \\ I, & \text{if } \theta = 0 \end{cases}$$

and I is the $n \times n$ identity matrix. Thereafter one must define the spectral projective operator P_0 corresponding to $\sigma_0 \subset \sigma(\mathbf{A})$. We will use its integral representation for that [4] in the form

$$(P_0\psi)(\theta) = \frac{1}{2\pi i} \int_{\mathbf{B}} ((Iz - \mathbf{A})^{-1}\psi)(\theta) dz \tag{4}$$

where $\mathbf{B} = \bigcup_{j=1}^{m} \{z : |z - z_j| = \delta\}$ with sufficiently small $\delta > 0$. It can be easily noticed that both projective operators P_0 and $I - P_0$ are bounded. Inserting the above matrix-valued function $\mathbf{I}$ into the integral representation (4) one can define the $n \times n$ matrix-function

$$\Gamma(\theta) = \frac{1}{2\pi i} \int_{\mathbf{B}} ((Iz - \mathbf{A})^{-1}\mathbf{I})(\theta) dz = \sum_{j=1}^{m} \operatorname{res}\{U^{-1}(z)e^{z\theta}\}\Big|_{z=z_j} \tag{5}$$

Let us denote the critical subspace as $\mathbf{X}_0 = P_0\mathbf{C}_n$, the matrix of a basis in this subspace (consisting of m columns and n rows) as $V(\theta)$ and the stricture of the operator $\mathbf{A}$ on $\mathbf{X}_0$ as $\mathbf{A}_0$ and let A_0 be the matrix of this stricture defined by the equation $\mathbf{A}_0 V(\theta) = V(\theta)A_0$. Furthermore, one can define the $m \times m$ matrix $\hat{\Psi}$ by the identity $\Gamma(\theta) = V(\theta)\hat{\Psi}$. Let us use the above notations along with the notation $\mathbf{V} = \{V(\theta), -h \le \theta \le 0\}$ and assume the existence of the $m-$ dimensional vector function $\breve{F}(x)$ of the argument $x \in \mathbf{R}^m$ defined by

$$\breve{F}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{\mathbf{Y}} e^{-tA_0} \hat{\Psi} F(t, \mathbf{V}e^{tA_0}x, y, 0)\mu(dy)dt .$$

Thus we define the averaged (not random) differential equation

$$\frac{d\breve{x}}{dt} = \breve{F}(x) . \tag{6}$$

II. Stability Theorems

By analogy with the corresponding definition of [5] we will say that the trivial solution of the random equation (1) is exponentially $p-$stable in the large for all sufficiently small positive ε if there exist positive constants ε_0, a_1 and positive number $a_2(\varepsilon)$ such that

$$\mathbf{E}_{y,\varphi}^{(s)} \left\{ \left| u^\varepsilon(t+s) \right|^p \right\} \le a_1 e^{-a_2(\varepsilon)t} \|\varphi\|^p$$

for any $s, t \ge 0$, $y \in \mathbf{Y}$ $\varphi \in \mathbf{C}_n$ and for any $\varepsilon \in (0, \varepsilon_0)$. Here and further throughout this paper the upper and lower indices of expectation denote the conditions $y(s) = y$, $u_s^\varepsilon = \varphi$.

Theorem 1. If the trivial solution of (2) is asymptotically stable, the trivial solution of the equation (1) is exponentially $p-$stable for any positive p and sufficiently small positive ε.

Theorem 2. Let the function $F(t, \mathbf{V}x, y, \varepsilon)$ be uniformly continuous at zero as a function of ε; have uniformly bounded continuous $x-$derivative $DF(t,\mathbf{V}x,y,0)$; belong to the domain $D(Q)$ of the operator Q; have continuous bounded $t-$derivative $\frac{\partial}{\partial t}F(t,\mathbf{V}x,y,0)$; have the above defined average $\breve{F}(x)$ along the solutions of the generative equation and let there exist constant b such that

$$\sup_{y,T,s}\left|\int_s^{s+T}\int_{\mathbf{Y}} e^{-tA_0}\hat{\Psi}F(t,\mathbf{V}e^{tA_0}x,y,0)\mu(dy)dt-T\breve{F}(x)\right|\le b|x|$$

for any $x\in\mathbf{R}^m$. Then if the trivial solution of the averaged equation (6) is globally exponentially stable, then the trivial solution of the random equation (1) is exponentially $p-$stable in the large for all sufficiently small, positive ε.

The proof is based on a projection in the critical subspace of the equation (3), transition to the slow time $\tau=\varepsilon t$ and applying the second Lyapunov method with a specially constructed Lyapunov functional as it has been done in [3] for equations with small Markov perturbations.

III. The Model and Results

Discussing a mathematical model of an adaptive Samuel-Marshall type single component market let us consider the equation

$$\frac{dp(t)}{dt}=D(p(t))-S(p(t-\tau))$$

where $D(p)$ and $S(p)$ denote, respectively, dependence of demand and supply on price p.

To control a price at the time moment t a manufacturer can use a supplied quantity *S(p(t)),* but to enter the market he/she needs some time τ *(t),* which may also be a stochastic process. Therefore a manufacturer is entering the market at the moments of time t and thus he/she has a delayed reaction because he/she is guided by the price at the moment of time *t* - τ *(t).* As a result, the supply depends on the price *p(t-*τ *(t)).* The demand at the time t instantaneously affects the price value, i.e., *D(p(t)).* Like in the classical Samuelson model we will suppose equilibrium to be reached due to an adaptive price dynamical property: the price movement *p(t +* τ *) - p(t)* is proportional to difference *D(p(t))- S(p(t-*τ *)).*

Our model supposes that a producer, having at his disposal resources to react on the increase in price, heightens supply immediately. In the opposite case the reaction is delayed by time τ, necessary for production or transportation of goods. Let us give a reasonable interpretation of time delay $\tau=y(t)$ as a Markov process with two states – zero and one. Let $y(t)$ be ergodic homogenous Markov process in space $\mathbf{Y}$ with transition probabilities $P(t,y,dz)$ and invariant measure $\mu\sim\{\pi,1-\pi\}$. Let us denote by $u(t)=p(t)-\bar{p}$ the price deviance from the equilibrium price $\bar{p}$. Let a and b indicate elasticity of demand and supply correspondingly. Using the ratio

$$c=\frac{a}{b}=\frac{D'(\bar{p})}{S'(\bar{p})}$$

and noting that both the supply and demand are non-linear functions, we may have the linearized model near the equilibrium price in the form

$$\frac{du(t)}{dt}=b(cu(t)-u(t-y(t)))+\varepsilon F(u_t,y(t)). \qquad (7)$$

By means of $G(\theta,y)=b(c\mathbf{1}_0(\theta)-\mathbf{1}_y(\theta))$, where

$$\mathbf{1}_\tau(\theta)=\begin{cases}0, & \theta\in[-1,0],\ \theta\notin\tau\\ 1, & \theta=\tau\end{cases},$$

one can rewrite the model in the form

$$\frac{du(t)}{dt}=\int_{-1}^{0}u(t+\theta)dG(\theta,y(t))+\varepsilon F(u_t,y(t)). \qquad (8)$$

To obtain the form useful for the averaging procedure we introduce operators:

$$(\mathbf{A}(y)\varphi)(\theta)=\begin{cases}\frac{d}{d\theta}\varphi(\theta), & -1\le\theta<0,\\ \int_{-1}^{0}\varphi(\theta)dG(\theta,y), & \theta=0.\end{cases}$$

That produces the equation in the space of continuous functions, $\mathbf{C}$:

$$\frac{d}{dt}u_t^\varepsilon=\mathbf{A}(y(t/\varepsilon))u_t^\varepsilon+\varepsilon\mathbf{I}F(t,u_t^\varepsilon,y(t)) \qquad (9)$$

Integration by the invariant measure μ provides

$$\frac{d}{dt}\bar{u}_t=\bar{\mathbf{A}}\bar{u}_t \qquad (10)$$

where

$$\bar{\mathbf{A}}\bar{u}_t=\int_{\mathbf{Y}}(\mathbf{A}(y)\bar{u}_t)(\theta)\mu(dy)=\begin{cases}\int_{-1}^{0}\bar{u}(t+\theta)d\bar{G}(\theta), & \theta=0,\\ \frac{d}{d\theta}\bar{u}(t+\theta), & -1\le\theta<0\end{cases}$$

$$\overline{G}(\theta) = \int_{\mathbf{Y}} G(\theta, y)\mu(dy).$$

Stability conditions depending on the spectrum of operator $\overline{\mathbf{A}}$

$$\sigma(\overline{\mathbf{A}}) = \{\det U(z) = 0\},$$

$$U(z) = I\,z - \int_{-1}^{0} e^{z\theta} d\overline{G}(\theta),$$

follow:

$$\overline{G}(\theta) = b(c-\pi)\mathbf{1}_0(\theta) - b(1-\pi)\mathbf{1}_{-1}(\theta),$$

$$\frac{d}{dt}\overline{u}(t) = b(c-\pi)\overline{u}(t) - b(1-\pi)\overline{u}(t-1),$$

$$\sigma(\overline{\mathbf{A}}) = \{z : z = b(c-\pi-(1-\pi)e^{-z})\}.$$

As $-1 < c < 0$, it is necessary to have $\pi < \frac{1-c}{2}$ and $b < \frac{\arccos((c-\pi)/(1-\pi))}{\sqrt{(1-c)(1+c-2\pi)}}$ to provide $\sigma(\overline{\mathbf{A}}) \subset \{\operatorname{Re} z < 0\}$.

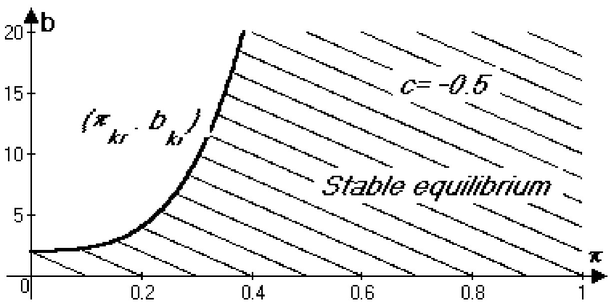

Fig. 1. Stability region for the market model

In case of $\sigma(\overline{\mathbf{A}}) \cap \{\operatorname{Re} z > 0\} = \varnothing$ and $\sigma(\overline{\mathbf{A}}) \cap \{\operatorname{Re} z = 0\} = \{z_1, z_2, ..., z_m\} \neq \varnothing$, i.e. the spectum of operator $\overline{\mathbf{A}}$ contains the set $\sigma_0 = \{z_1, z_2, ..., z_m\}$ of m imaginary points one may use the results of Theorem 2.

Let us specify the demand and supply functions for the model to provide the case if the system is on the border of stability:

$$D(p) = ap + \varepsilon\gamma p^3 + \alpha$$

$$S(p) = bp + \beta$$

$$\pi = \pi_{kr} - \varepsilon\gamma$$

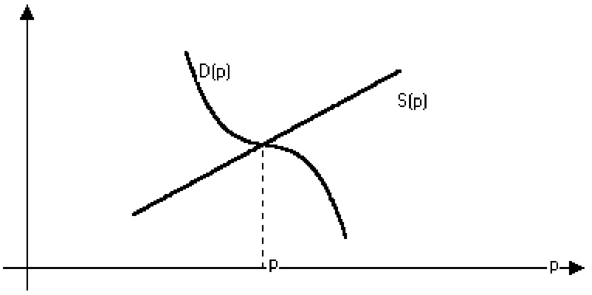

Fig. 2. Specified demand and supply functions for obtaining business cycle

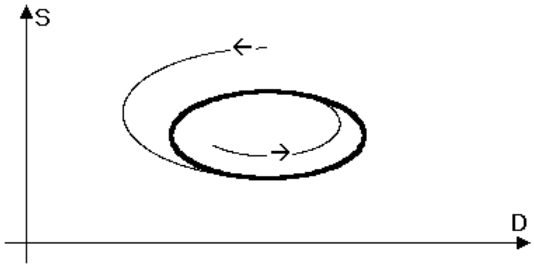

Fig. 3. Business cycle on the demand-supply plane

Then operator $\overline{\mathbf{A}}$ has a pair of imaginary eigenvalues $\sigma_0 = \{\pm i\nu\}$ with $\nu = \sqrt{(1-c)(1+c-2\pi_{kr})}$. Analysis of the averaged system allows establishing existence of the stable phase trajectories business cycle on the (D, S) plane for the system (7).

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Vineta Minkēviča, Kārlis Šadurskis. Par adaptīva vienas komponentes tirgus līdzsvaru
Jebkuras reālas sistēmas vai parādības attīstība lielākā vai mazākā mērā pakļauta nenoteiktības iedarbībai, pie kam sistēmas parametri gadījuma ietekmē var mainīties samērā vienmērīgi vai lēcienveidīgi. Interesi izraisa šāda veida sistēmu līdzsvara nosacījumu izpēte. Ekonomiskās sfēras procesu modelēšanai, kā parasti, tiek izmantoti determinēti bāzes modeļi, kuros sākotnēji gadījuma ietekme netiek ņemta vērā. Dotajā rakstā apskatītajā klasiskajā adaptīva vienas komponentes Semjuela-Māršala tipa tirgus modelī pieprasījums un piedāvājums tiek definēti kā cenas funkcijas D(p) un S(p), cenai mainoties atkarībā no laika p(t). Modelis paredz, ka ražotāja reakcijai uz tirgus cenu nepieciešams laiks τ, līdz ar to ražotāja reakcija aizkavējas, jo viņš rīkojas atbilstoši cenai momentā t-τ un atbildot uz pieprasījuma vērtību D(p(t)) piedāvā apjomu S(p(t-τ)). Semjuela-Māršala modelī tirgus reakcijas cenu starpība ir proporcionāla atbilstošā pieprasījuma un piedāvājuma starpībai D_t – S_t. Saglabājot šo modeļa pamatideju, rakstā tiek piedāvāts apskatīt ražotāja reakcijai nepieciešamo laiku τ kā stohastisku procesu, kas attīstās lēcienveidīgi. Šāda modeļa aprakstam tiek piedāvāts kvazilineārs funkcionāldiferenciālvienādojums, kurš atkarīgs no fāzes koordinātas un ergodiska Markova procesa ar ātriem pārslēgumiem. Raksta pamatdaļa veltīta apskatāmās sistēmas līdzsvara nosacījumu izpētei. Stohastiskā modeļa stabilitātes novērtējumam tiek piedāvāta metode, kas balstās uz vidējošanas procedūru attiecībā uz laiku, sekojot ģeneratīva vidējā lineārā vienādojuma kritiskajiem risinājumiem, un attiecībā uz Markova procesa invarianto mēru. Pielietojot minēto metodi pētāmajam modelim, tiek pierādīts, ka rezultējošā determinētā vienādojuma eksponenciālā stabilitāte ir pietiekama sākotnējās nenoteiktās, gadījuma iedarbībai pakļautās sistēmas eksponenciālajai stabilitātei, pietiekami ātru pārslēgumu gadījumā.

Винета Минкевича, Карлис Шадурскис. О равновесии адаптивного однокомпонентного рынка
Развитие любой реальной системы или явления происходит под влиянием неопределенности. При этом параметры системы под воздействием случая меняются более или менее равномерно или скачкообразно. Интерес вызывает исследование условий равновесия таких систем. Для моделирования процессов экономической сферы, как правило, используются детерминированные базовые модели, в которых первоначально не учитываются случайные эффекты. В классической модели однокомпонентного адаптивного рынка типа Сэмюэла-Маршалла, которая рассматривается в данной статье, спрос и предложение определяются как функции цены - D (p) и S (p), а цена меняется в зависимости от времени - p (t). Модель предусматривает, что ответ производителя на изменение рыночной цены требует времени τ, поэтому реакция производителя задерживается, так как он действует в соответствии с ценой в момент времени t- τ, и в ответ на значение спроса D(p(t)) предоставляет количество S(p(t-τ)). В модели Сэмюэла-Маршалла разность цен ответной реакции рынка пропорциональна разности спроса и предложения D_t – S_t. Сохраняя основную идею этой модели, в статье предложено время на реакцию производителя расссматривать как стохастическй процесс, который развивается скачкообразно. Для описания такой системы предлагается использовать квазилинейные функциональные уравнения, зависящие от координаты фазы и эргодического процесса Маркова с быстрыми переключениями. Основная часть статьи посвящена исследованию условий равновесия рассматриваемой системы. Для оценки стабильности стохастической модели предлагается метод, который основан на процедуре усреднения относительно времени вдоль критических решений порождающего среднего линейного уравнения и относительно инвариантной меры Марковского процесса. С помощью данного метода доказано, что экспоненциальная стабильность полученного детерминированного уравнения достаточна для экспоненциальной стабильности начальной случайной системы для всех положительных чисел и для достаточно быстрых переключений.