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One Method for Determination of Thermal and Physical Characteristics of Film-Base Materials

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Abstract **- There is a calculation method for evaluating thermal and physical characteristics of thin materials (temperature conductivity and thermal conductivity coefficient) by using temperature measurements given in this article. Mathematical reasoning of the method is also provided in the paper. Implementation of this method is discussed by using temperature field as input information obtained by means of the software MATHEMATICA.**

Keywords **- derivative, series, temperature field, temperature conductivity coefficient.**

I. INTRODUCTION

The majority of determination methods of thermal and physical characteristics is based on the fact that temperature measurements are made in a solid of simple shape, readings of such measurements are compared with a mathematic model of heat transfer, and characteristics under discussion are calculated afterwards. If some material is thin or plate-shaped, for example, film, paper, window glass, then temperature measurements inside such material are impossible as dimensions of thermocouple are comparable to the ones of material. In such a case we offer to put the material under examination between two plates of other material, whose thermal and physical characteristics are known, and make temperature measurements in those plates. Measurement scheme is given in Fig. 1, where the material under investigation is located in the area *[0,b]*. In the area *[-l,0]* and *[b,l+b]*, (*l>>.b*) the material is located whose temperature conductivity and thermal conductivity coefficients are known as a_1 *and* λ_1 respectively. Non-stationary thermal conductivity process is ensured. Temperature is measured at four points: x_1, x_2, x_3, *and x4*.

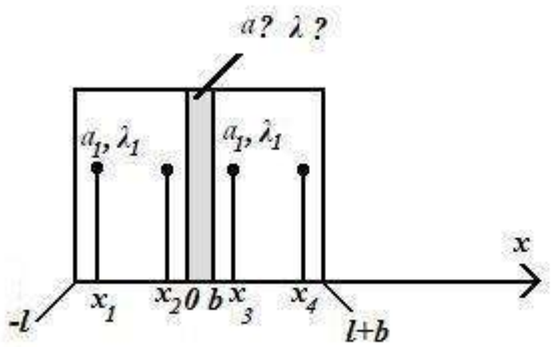

Fig. 1 Detection scheme of the heat physical characteristics.

The problem here is to determine temperature conductivity coefficient *a* of the material under examination and thermal conductivity coefficient λ by using temperature measurements at those points. At the measurement point of both external thermocouples, temperature measurements may also be made on the external boundaries if such are possible.

II. MATHEMATIC MODEL

It is obvious that the process of thermal conductivity in plates illustrated in Figure 1 can be described with the following equations:

$$\frac{\partial t}{\partial \tau} = a_1 \frac{\partial^2 t}{\partial x^2}, \; x \in [-l, 0] \quad (1)$$

$$\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial x^2}, \; x \in [0, b] \quad (2)$$

$$\frac{\partial t}{\partial \tau} = a_1 \frac{\partial^2 t}{\partial x^2}, \; x \in [b, l + b]. \quad (3)$$

Supposing that temperature distribution is even at the beginning $t(x,0) = t_0$, let us displace the temperature measurements for this value so that the following would be satisfied:

$$t(x,0) = 0 \quad (4)$$

The instance will be considered when boundary conditions of the first kind are given on the external boundaries, i.e.,

$$t(-l, \tau) = f_1(\tau), t(l + b, \tau) = f_2(\tau) \quad (5)$$

For problem (1)–(5) to be formulated unequivocally, the following conditions should be added obviously

$$t(-0, \tau) = t(+0, \tau), \; t(b-0,\tau) = t(b+0,\tau). \quad (6)$$

Problem (1)–(6) with different input information has been solved numerically by means of the software MATHEMATICA. Temperature values at coordinates have been obtained from the solution to the problem complying with the location of thermocouples, and an inverse problem is solved with such input information.

III. Solution to the Inverse Problem

In accordance with [1], the solution to the thermal conductivity equation (1) with boundary conditions of the first kind and homogenous initial conditions at interval $x \in [x_1, x_2]$ and expressed by non-dimensional variables is as follows:

$$t(N,F) = \sum_{n=0}^{\infty} \left(t_1^{(n)}(F) P_n(N,0) + t_2^{(n)}(F) P_n(N,1) \right), \quad (7)$$

where $N = \dfrac{x - x_1}{x_2 - x_1}$, $N \in [0,1]$, $F = \dfrac{a_1 \tau}{(x_2 - x_1)^2}$ are non-dimensional coordinate and non-dimensional time respectively, $t_1(F)$ and $t_2(F)$ – temperature at $x=x_1$ and $x=x_2$ respectively, $P_n(N,0)$ and $P_n(N,1)$ are functions of coordinate, which are pursuant to [1]:

$$P_0(N,0) = -N + 1,$$

$$P_1(N,0) = -\frac{1}{6}N^3 + \frac{1}{2}N^2 - \frac{1}{3}N,$$

$$P_2(N,0) = -\frac{1}{120}N^5 + \frac{1}{24}N^4 - \frac{1}{18}N^3 + \frac{1}{45}N,$$

$$P_3(N,0) = -\frac{1}{5040}N^7 + \frac{1}{720}N^6 - \frac{1}{360}N^5 + \frac{1}{270}N^3 - \frac{2}{945}N,$$

$$P_4(N,0) = -\frac{1}{362880}N^9 + \frac{1}{40320}N^8 - \frac{1}{15720}N^7 + \frac{1}{5040}N^5 - -\frac{1}{2835}N^3 + \frac{1}{5040}N,$$

$$P_0(N,1) = N,$$

$$P_1(N,1) = \frac{1}{6}N^3 - \frac{1}{6}N,$$

$$P_2(N,1) = \frac{1}{120}N^5 - \frac{1}{36}N^3 + \frac{7}{360}N,$$

$$P_3(N,1) = \frac{1}{720}\left(\frac{1}{7}N^9 - N^5 + \frac{7}{3}N^3 - \frac{31}{21}N \right),$$

$$P_4(N,1) = \frac{1}{30240}\left(\frac{1}{12}N^9 - N^7 + \frac{49}{10}N^5 - \frac{31}{3}N^3 + \frac{127}{20}N \right).$$

Under this coordinate system, the non-dimensional coordinate $N_1 = \dfrac{-x_1}{x_2 - x_1} > 1$ complies with a point $x=0$. Temperature field interpretation in the shape (7) for various geometries and boundary conditions is the most widely discussed topic in the paper [1]. It is also used in many other papers, for instance, [2] and [3]. The paper [1] proved that temperature at $x=0$ is expressed by the formula (7) if $N=N_1$ is inserted therein, consequently:

$$t(0,F) = \sum_{n=0}^{\infty} \left(t_1^{(n)}(F) P_n(N_1,0) + t_2^{(n)}(F) P_n(N_1,1) \right) \quad (8)$$

Solution to the equation (3) in the interval $x \in [x_3, x_4]$ is

$$t(N,F) = \sum_{n=0}^{\infty} \left(t_3^{(n)}(F) P_n(N,0) + t_4^{(n)}(F) P_n(N,1) \right), \quad (9)$$

where $N = \dfrac{x - x_3}{x_4 - x_3}$, $N \in [0,1]$, $F = \dfrac{a_1 \tau}{(x_4 - x_3)^2}$ are non-dimensional coordinate and non-dimensional time respectively, $t_3(F)$, and $t_4(F)$ – temperature at $x=x_3$ and $x=x_4$ respectively.

Under this coordinate system, non-dimensional coordinate $N_2 = \dfrac{b - x_3}{x_4 - x_3} < 0$ complies with a point $x=b$. Analogously to the previous instance, temperature at $x=b$ is expressed as follows:

$$t(b,F) = \sum_{n=0}^{\infty} \left(t_3^{(n)}(F) P_n(N_2,0) + t_4^{(n)}(F) P_n(N_2,1) \right). \quad (10)$$

Further let us presume that equality $x_4 - x_3 = x_2 - x_1 = b_1$ is satisfied. By replacing non-dimensional time F by τ in the formulae (8) and (10), we obtain:

$$t(0,\tau) = \sum_{n=0}^{\infty} \left(t_1^{(n)}(\tau) P_n(N_1,0) + t_2^{(n)}(\tau) P_n(N_1,1) \right) \left(\frac{b_1^2}{a_1} \right)^n \quad (11)$$

$$t(b,\tau) = \sum_{n=0}^{\infty} \left(t_3^{(n)}(\tau) P_n(N_2,0) + t_4^{(n)}(\tau) P_n(N_2,1) \right) \left(\frac{b_1^2}{a_1} \right)^n \quad (12)$$

Heat flow at $x=0$ is equal to

$$q(0,\tau) = -\lambda_1 \frac{\partial t(0,\tau)}{\partial x} = = -\frac{\lambda_1}{b_1} \sum_{n=0}^{\infty} \left(t_1^{(n)}(\tau) P_n'(N_1,0) + t_2^{(n)}(\tau) P_n'(N_1,1) \right) \left(\frac{b_1^2}{a_1} \right)^n \quad (13)$$

Heat flow at $x=b$ equals to

$$q(b,\tau)=-\lambda_1\frac{\partial t(b,\tau)}{\partial x}=$$
$$=-\frac{\lambda_1}{b_1}\sum_{n=0}^{\infty}\left(t_3^{(n)}(\tau)P_n'(N_2,0)+t_4^{(n)}(\tau)P_n'(N_2,1)\right)\left(\frac{b_1^2}{a_1}\right)^n \quad (14)$$

Ratio of heat flows is a calculable value:

$$\frac{q(0,\tau)}{q(b,\tau)}=\frac{\sum_{n=0}^{\infty}\left(t_1^{(n)}(\tau)P_n'(N_1,0)+t_2^{(n)}(\tau)P_n'(N_1,1)\right)\left(\frac{b_1^2}{a_1}\right)^n}{\sum_{n=0}^{\infty}\left(t_3^{(n)}(\tau)P_n'(N_2,0)+t_4^{(n)}(\tau)P_n'(N_2,1)\right)\left(\frac{b_1^2}{a_1}\right)^n}. \quad (15)$$

It follows from (7) that the temperature field in the material under examination when $x\in[0,b]$ may be written down as

$$t(x,\tau)=\sum_{n=0}^{\infty}\left(t^{(n)}(0,\tau)P_n(N,0)+t^{(n)}(b,\tau)P_n(N,1)\right)\left(\frac{b^2}{a}\right)^n, \quad (16)$$

where $N=x/b$, $N\in[0,1]$.

This temperature cannot be calculated because it depends on the unknown temperature conductivity coefficient a. Heat flow at $x=0$ and $x=b$, i.e., $q(0,\tau)$ and $q(b,\tau)$, is also defined by means of the formula (16):

$$q(0,\tau)=-\lambda\frac{\partial t(+0,\tau)}{\partial x}=$$
$$=-\frac{\lambda}{b}\sum_{n=0}^{\infty}\left(t^{(n)}(0,\tau)P_n'(0,0)+t^{(n)}(b,\tau)P_n'(0,1)\right)\left(\frac{b^2}{a}\right)^n \quad (17)$$

$$q(b,\tau)=-\lambda\frac{\partial t(b-0,\tau)}{\partial x}=$$
$$=-\frac{\lambda}{b}\sum_{n=0}^{\infty}\left(t^{(n)}(0,\tau)P_n'(1,0)+t^{(n)}(b,\tau)P_n'(1,1)\right)\left(\frac{b^2}{a}\right)^n \quad (18)$$

By denoting $\frac{q(0,\tau)}{q(b,\tau)}=k$, $\frac{b^2}{a}=z$, $t(0,\tau)=T_0$, $t(b,\tau)=T_1$, and taking only two addends in the formulae (17) and (18), the following is obtained:

$$k=\frac{T_0P_0'(0,0)+T_1P_0'(0,1)+\left(T_0'P_1'(0,0)+T_1'P_1'(0,1)\right)z}{T_0P_0'(1,0)+T_1P_0'(1,1)+\left(T_0'P_1'(1,0)+T_1'P_1'(1,1)\right)z}. \quad (19)$$

By expressing z from the last equality and taking into consideration previously given expressions of coordinate functions, we obtain

$$z=\frac{6(k-1)(T_0-T_1)}{(2+k)T_0'+(1+2k)T_1'}, \quad (20)$$

or

$$a=\frac{b^2\left((2+k)T_0'+(1+2k)T_1'\right)}{6(k-1)(T_0-T_1)}. \quad (21)$$

Taking three addends in the formulae (17) and (18), the following equation is obtained:

$$k=\frac{T_0P_0'(0,0)+T_1P_0'(0,1)+\left(T_0'P_1'(0,0)+T_1'P_1'(0,1)\right)z+\left(T_0''P_2'(0,0)+T_1'(0,1)P_2'(0,1)\right)z^2}{T_0P_0'(1,0)+T_1P_0'(1,1)+\left(T_0'P_1'(1,0)+T_1'P_1'(1,1)\right)z+\left(T_0''P_2'(1,0)+T_1''P_2'(1,1)\right)z^2}.$$

Taking into account previously given expressions of coordinate functions, the last equation may be re-written as follows:

$$\left(-T_0+T_1+\left(\frac{1}{6}T_0'+\frac{1}{3}T_1'\right)z+\left(\frac{-7}{360}T_0''-\frac{1}{45}T_1''\right)z^2\right)k=$$
$$=-T_0+T_1-\left(\frac{1}{3}T_0'+\frac{1}{6}T_1'\right)z+\left(\frac{1}{45}T_0''+\frac{7}{360}T_1''\right)z^2 \quad (22)$$

It follows that

$$a=\frac{b^2}{z} \quad (23)$$

Equation (22) is a quadratic equation and has two roots. We have not proved, but practical calculations at the whole spectrum of conditions showed that one root always is positive whereas the second one is negative.

Thermal conductivity coefficient may be determined from heat flow equality on borders of the material under examination. It results from flow equality on the left border that

$$\lambda=\lambda_1\frac{\partial t(-0,\tau)}{\partial x}/\frac{\partial t(+0,\tau)}{\partial x}, \quad (24)$$

where $\lambda_1\frac{\partial t(-0,\tau)}{\partial x}$ is calculated by means of the formula (13). Deriving the formula (16) as per x and inserting the obtained temperature conductivity coefficient and $N=0$, we obtain

$$\frac{\partial t(+0,\tau)}{\partial x}=\frac{1}{b}\sum_{n=0}^{\infty}\left(t^{(n)}(0,\tau)P_n'(0,0)+t^{(n)}(b,\tau)P_n'(0,1)\right)\left(\frac{b^2}{a}\right)^n$$

IV. NUMERICAL EXAMPLE

The above-discussed scheme may be used for determination of temperature conductivity coefficient of thin material that complies with Fig. 1. Input information:

$$l=0.02 \text{ m},\ b=0.002 \text{ m},\ a_1=10^{-4}\ \frac{m^2}{s},\ a=10^{-6}\ \frac{m^2}{s}.$$

There are the following boundary conditions (5) given: $t(-l,\tau)=100-100e^{-0.001\tau}$, $t(b+l,\tau)=0$.

By means of the software MATHEMATICA under such conditions, problem (1)–(6) has been solved. Its solution is showed in Fig. 2.

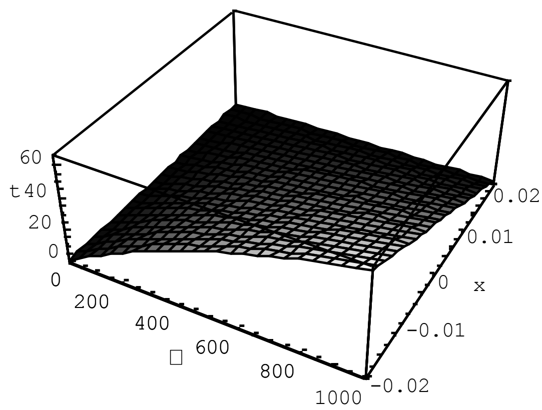

Fig. 2. Temperature field

Temperature change in time $t(x_i,\tau)$ is obtained from numerical solution at $x_1=-0.014$, $x_2=-0.006$, $x_3=0.008$, $x_4=0.016$. Corresponding charts are given in Fig.3. Values of this temperature are used as input information of the inverse problem. The aim of the inverse problem is to determine the temperature conductivity coefficient, from the data in the area $x\in[0,b]$.

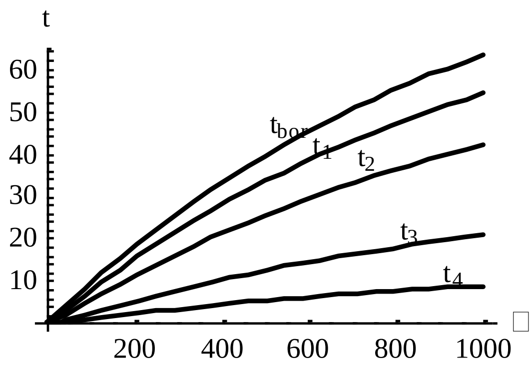

Fig. 3. tbor=t(-l,τ), ti=t(xi,τ), i=1,2,3,4

Temperatures calculated by means of the formulae (11), (12) at $x=0$ and $x=b$ are provided in Fig.4.

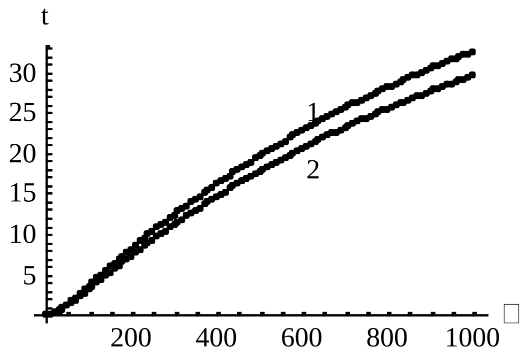

Fig. 4. Curve 1 – t(0,τ), curve 2 – t(b,τ)

Difference between the solution to the problem (1)–(6) at a point $x=0$ and the temperature values obtained from the formula (11) at this point is showed in Fig. 5. Situation is similar at a point $x=b$.

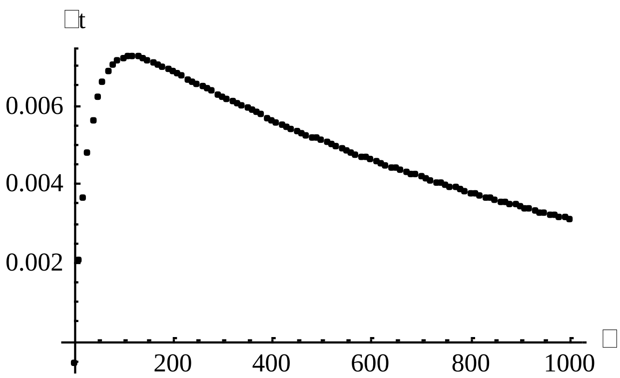

Fig. 5. Difference of temperatures

Flow ratio calculated by means of the formula (15) is given in Fig. 6.

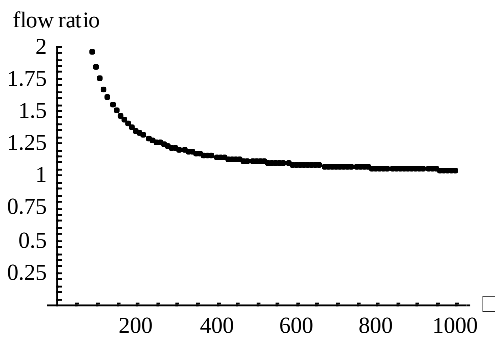

Fig 6. Flow ratio

Temperature conductivity coefficient has been calculated by using the formula (21) in moments of time with even interval $\tau=50,60,\dots,1000$. Average value of temperature conductivity coefficient is $a=1.9940\cdot10^{-6}\ \frac{m^2}{s}$, relative error expressed as percentage – *0.5996*, standard deviation of temperature conductivity coefficient – $9.4416\cdot10^{-10}$. Temperature conductivity coefficient has been calculated by using the formulae (22), (23) in moments of time with even interval $\tau=80, 90,\ldots,1000$. Average value of temperature conductivity coefficient is $a=0.9946\cdot 10^{-6}\ \frac{m^2}{s}$, relative error expressed as percentage – *0.5353*, standard deviation of temperature conductivity coefficient – $8.0439\cdot 10^{-10}$.

V. Conclusion

Many calculations have been performed under various boundary conditions and different geometric dimensions. The results have showed some instability at little values of time. Therefore these results do not include results of calculations at little values of time. Formula (22) should have been more precise than the formula (21) because three addends of series are taken in the formula (22). Moreover, instability of results is even larger when using the formula (22) than instability of results when using the formula (21). Thus, we would recommend applying the formula (21).

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Ilmārs Iltiņš, Marija Iltiņa. Viena metode siltumfizikālo raksturlielumu noteikšanai plēvjveida materiāliem
Lielākā daļa no cieta ķermeņa siltumfizikālo raksturlielumu (temperatūrvadīšanas koeficients, siltumvadīšanas koeficients) noteikšanas metodēm balstās uz temperatūras mērījumiem vienkāršas formas ķermeņa iekšienē nestacionāra siltumvadīšanas procesa laikā. Ja materiāls ir plāns (tā izmēri būtiski nepārsniedz termopāra izmērus), tad šādi mērījumi nav iespējami. Rakstā tiek piedāvāts pētāmo materiālu ievietot starp diviem plāksnes veida materiāliem ar zināmiem siltumfizikālajiem raksturlielumiem un temperatūras mērījumus veikt šajos materiālos četrās vietās. Izmantojot šos mērījumus, vispirms tiek noteikta temperatūra un siltuma plūsma uz pētāmā materiāla robežas. Tas apstāklis, ka temperatūras lauka noteikšanai pētāmajā materiālā, pietiek tikai ar vienu no šiem lielumiem, dod iespēju formulēt inverso uzdevumu attiecībā uz pētāmā materiāla temperatūrvadīšanas koeficientu. Lai pārbaudītu piedāvātās metodes pielietojamību, temperatūras lauks trijās blakus novietotās plāksnēs ar atšķirīgiem siltumfizikālajiem raksturlielumiem tiek iegūts ar datorprogrammas MATHEMATICA palīdzību. Uzdevums tiek risināts, par ieejas datiem izmantojot temperatūras vērtības četros punktos. Kā temperatūras lauka matemātiskais modelis tiek izmantots tā izvirzījums rindā pa pirmā veida robežnosacījumu atvasinājumiem. Šāds matemātiskais modelis noved pie vienkāršām temperatūrvadīšanas un siltumvadīšanas koeficientu aprēķina formulām.

Илмарс Илтиньш, Мария Илтиня. Один из методов определения теплофизических характеристик пленочных материалов
Большинство методов определения теплофизических характеристик твердого тела (коэффициент теплопроводности и коэффициент температуропроводности) основаны на измерении при нестационарном процессе теплопроводности температуры внутри тела, имеющего простую форму. Если материал тонкий (размер не значительно превышает размеры термопары), то такие измерения невозможны. В статье предложено поместить исследуемый материал между двумя пластинами с известными теплофизическими характеристиками. Измерения температуры проводятся в пластинах в четырех местах. С помощью этих измерений сначала определяется температура и тепловой поток на границе исследуемого материала. Тот факт, что для определения температурного поля исследуемого материала достаточно только одного из этих параметров, приводит к возможности сформулировать обратную задачу относительно коэффициента температуропроводности материала. Для проверки применимости предлагаемого метода, температурное поле в трех, рядом помещенных, пластинах с различными теплофизическими характеристиками, получено с помощью компьютерной программы MATHEMATICA. Задача решена с использованием значений температуры в четырех точках. В качестве математической модели температурного поля используется его разложение в ряд по производным от граничных условий первого рода. Такая математическая модель приводит к простым формулам для вычисления коэффициентов теплопроводности и температуропроводности.