Analytical Solution to an Eddy Current Testing Problem for a Cylindrical Tube with Varying Properties

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Abstract **– Closed-form solution for the change in impedance of a single-turn coil located inside a conducting cylindrical tube is presented in the paper. The axis of the coil coincides with the axis of the tube. The electric conductivity and magnetic permeability of the tube are power functions of the radial coordinate. The system of equations for the vector potential is solved by the method of Fourier cosine transform. Equations for the transformed components of the vector potential in free space are solved in terms of the modified Bessel functions of order one. The ordinary differential equation for the transformed component of the vector potential in the tube is solved by means of the confluent hypergeometric function. Results of numerical calculations are presented in the paper.**

Keywords **– eddy current testing, electric conductivity, Fourier transform, confluent hypergeometric function, magnetic permeability**

I. INTRODUCTION

Theory for eddy current testing problems for multilayer cylindrical regions with constant properties is well developed in the literature [1]-[3]. For some engineering applications such as surface hardening and de-carbonization, it is necessary to develop a theory for the case where the electrical conductivity and magnetic permeability of a conducting medium are not constants (see, for example, [4] and [5]). Analytical solutions to eddy current testing problems for multilayer cylindrical tubes are presented in [6] in terms of modified Bessel functions for the case where the electrical conductivity and magnetic permeability are power functions of the radial coordinate.

In the present paper we construct a one-parameter family of analytical solutions of eddy current problem for a conducting tube where the properties of the tube are modelled by power functions of the radial coordinate. The solution is expressed in terms of improper integrals containing confluent hypergeometric functions and modified Bessel functions.

II. MATHEMATICAL FORMULATION

Suppose that a single-turn coil of radius r_c is located inside an infinitely long cylindrical tube of the inner and outer radii r_1 and r_2, respectively. The centre of the coil is situated at the plane $z = 0$. In addition, the axis of the coil coincides with the axis of the tube. The inner radius of the tube r_1 is chosen as the measure of length. The dimensionless outer radius of the tube is R.

Consider the following three regions:

(a) the free space

$$R_0 : 0 \le r \le 1, 0 \le \varphi \le 2\pi, -\infty < z < +\infty;$$

(b) the cylindrical layer

$$R_1 : 1 \le r \le R, 0 \le \varphi \le 2\pi, -\infty < z < +\infty;$$

and the outer free space

$$R_2 : R < r, 0 \le \varphi \le 2\pi, -\infty < z < +\infty.$$

The current in the coil is given by the formula

$$i(t)\vec{e}_\varphi = I \exp(j\omega t)\vec{e}_\varphi, \tag{1}$$

where I is the amplitude of the current, ω is the frequency and $\vec{e}_\varphi$ is the unit vector in the $\varphi-$ direction.

We assume that the magnetic permeability μ and electric conductivity σ are power functions of the radial coordinate of the form

$$\mu = \mu_0 \mu_* r^{\alpha}, \ \sigma = \sigma_* r^{\beta}, \tag{2}$$

where α, β, μ_* and σ_* are given constants.

The vector potential $\hat{A}_i, i = 0,1,2$ is represented in the form

$$\hat{A}_i(r,z,t)\vec{e}_\varphi = A_i(r,z)\exp(j\omega t)\vec{e}_\varphi. \tag{3}$$

Using (1)-(3), we obtain the following system of equations for the amplitudes of the vector potential in regions R_0, R_1 and R_2 (see [6]):

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r}\frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 r_1{}^2 I \delta(r - r_0)\delta(z), \tag{4}$$

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{(1-\alpha)}{r}\frac{\partial A_1}{\partial r} - \left(\frac{1+\alpha}{r^2} + p^2 r^{\alpha+\beta}\right)A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0, \tag{5}$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r}\frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} + \frac{\partial^2 A_2}{\partial z^2} = 0, \tag{6}$$

where $p = r_c\sqrt{j\omega\sigma\mu_0}$, $r_0 = r_c / r_1$ and $\delta(x)$ is the Dirac delta-function.

The boundary conditions are

$$A_0 |_{r=1} = A_1 |_{r=1}, \ \frac{\partial A_0}{\partial r} |_{r=1} = \frac{1}{\mu_1}\frac{\partial A_1}{\partial r} |_{r=1}, \tag{7}$$

$$A_1 |_{r=R} = A_2 |_{r=R}, \ \frac{1}{\mu_2}\frac{\partial A_1}{\partial r} |_{r=R} = \frac{\partial A_2}{\partial r} |_{r=R},. \tag{8}$$

where $\mu_1 = \mu(r_1)$ and $\mu_2 = \mu(r_2)$.

In addition, A_0 is bounded as $r \to 0$ and A_2 is bounded as $r \to \infty$.

III. SOLUTION TO THE PROBLEM

Applying the Fourier cosine transform of the form

$$\tilde{A}_i(r,\lambda) = \int_0^\infty A_i(r,z)\cos\lambda z dz, \quad i = 0,1,2, \tag{9}$$

to (4)-(8) we obtain

$$\frac{d^2\tilde{A}_0}{dr^2} + \frac{1}{r}\frac{d\tilde{A}_0}{dr} - \frac{\tilde{A}_0}{r^2} - \lambda^2\tilde{A}_0 = -\mu_0\frac{I r_1^2}{2}\delta(r - r_0), \tag{10}$$

$$\frac{d^2\tilde{A}_1}{dr^2} + \frac{(1-\alpha)}{r}\frac{d\tilde{A}_1}{dr} - \left(\frac{1+\alpha}{r^2} + p^2 r^{\alpha+\beta} + \lambda^2\right)\tilde{A}_1 = 0, \tag{11}$$

$$\frac{d^2\tilde{A}_2}{dr^2} + \frac{1}{r}\frac{d\tilde{A}_2}{dr} - \frac{\tilde{A}_2}{r^2} - \lambda^2\tilde{A}_2 = 0. \tag{12}$$

The boundary conditions have the form

$$\tilde{A}_0 |_{r=1} = \tilde{A}_1 |_{r=1}, \ \frac{d\tilde{A}_0}{dr} |_{r=1} = \frac{1}{\mu_1}\frac{d\tilde{A}_1}{dr} |_{r=1}, \tag{13}$$

$$\tilde{A}_1 |_{r=R} = \tilde{A}_2 |_{r=R}, \ \frac{1}{\mu_2}\frac{d\tilde{A}_1}{dr} |_{r=R} = \frac{d\tilde{A}_2}{dr} |_{r=R} . \tag{14}$$

Some analytical solutions of (11) for different values of the parameters α and β in terms of the modified Bessel functions are presented in [6]. Here we present a one-parameter family of analytical solutions of (11) for the case $\alpha + \beta = -1$. Using the substitution

$$\tilde{A}_1 = r^{(\alpha-1)/2}\tilde{B} \tag{15}$$

we transform (11) to the Whittaker's equation (see [7])

$$\tilde{B}_{xx} + \left(-\frac{1}{4} + \frac{k}{x} + \frac{1/4 - \nu^2}{x^2}\right)\tilde{B} = 0, \tag{16}$$

where $x = 2\lambda r,\ k = -\dfrac{p^2}{2\lambda},\ \nu = \dfrac{\alpha}{2} + 1.$

Using the substitution

$$\tilde{B} = e^{x/2}x^{-c/2}y \tag{17}$$

equation (16) can be written in the form

$$y'' + \left(\frac{c}{x} - 1\right)y' - \frac{a}{x}y = 0, \tag{18}$$

where $a = \nu - k + 1/2$ and $c = 2\nu + 1$.

General solution to (18) can be expressed in the form:

$$y = C_4\psi(a,c,x) + C_5 e^x\psi(c-a,c,x), \tag{19}$$

where $\psi(a,c,x)$ is the confluent hypergeometric function (see [7]), and C_4, C_5 are arbitrary constants. Thus, a general solution to (11) can be written in the form

$$\begin{aligned}\tilde{A}_1 &= C_4 r^{\alpha+1}e^{-\lambda r}\psi(a,c,2\lambda r)\\ &+ C_5 r^{\alpha+1}e^{\lambda r}\psi(c-a,c,-2\lambda r).\end{aligned} \tag{20}$$

In order to simplify the notations we define the following functions

$$\varphi_1(r) = r^{\alpha+1}e^{-\lambda r}\psi(a,c,2\lambda r)$$

and

$$\varphi_2(r) = r^{\alpha+1}e^{\lambda r}\psi(c-a,c,-2\lambda r).$$

Hence,

$$\tilde{A}_1 = C_4\varphi_1(r) + C_5\varphi_2(r). \tag{21}$$

The solution to (10) is found in the following two sub-regions of R_0, namely,

$$R_{00} = \{0 \le r < r_0, 0 \le \varphi \le 2\pi, -\infty < z < +\infty\}$$

and

$$R_{01} = \{r_0 < r < 1, 0 \le \varphi \le 2\pi, -\infty < z < +\infty\}.$$

The bounded solution to (10) in region R_{00} has the form

$$\tilde{A}_{00} = C_1 I_1(\lambda r), \tag{22}$$

where $I_1(\lambda r)$ is the modified Bessel function of the first kind of order 1.

The general solution to (10) in region R_{01} is

$$\tilde{A}_{01} = C_2 I_1(\lambda r) + C_3 K_1(\lambda r), \tag{23}$$

where $K_1(\lambda r)$ is the modified Bessel function of the second kind of order 1.

The functions $\tilde{A}_{00}$ and $\tilde{A}_{01}$ satisfy the following conditions at $r = r_0$:

$$\tilde{A}_{00} |_{r=r_0} = \tilde{A}_{01} |_{r=r_0}, \tag{24}$$

$$\frac{d\tilde{A}_{01}}{dr} |_{r=r_0} - \frac{d\tilde{A}_{00}}{dr} |_{r=r_0} = -\mu_0 \frac{Ir_1^2}{2}. \tag{25}$$

The bounded solution to (12) is

$$\tilde{A}_2 = C_6 K_1(\lambda r). \tag{26}$$

The coefficients $C_1 - C_6$ in (21)-(23), (26) are obtained from the boundary conditions (13), (14), (24), (25). In particular,

$$C_2 = -\frac{\mu_0 I r_1^2}{2} r_0 I_1(\lambda r_0) \frac{D}{E}, \tag{27}$$

where

$$D = K_1(\lambda)[\gamma\varphi_1'(1) - \varphi_2'(1)] + \lambda\mu_1 K_1'(\lambda)[\varphi_2(1) - \gamma\varphi_1(1)],$$

$$E = I_1(\lambda)[\gamma\varphi_1'(1) - \varphi_2'(1)] + \lambda\mu_1 I_1'(\lambda)[\varphi_2(1) - \gamma\varphi_1(1)],$$

$$\gamma = \frac{\mu_2 \lambda K_1'(\lambda R)\varphi_2(R) - K_1(\lambda R)\varphi_2'(R)}{\mu_2 \lambda K_1'(\lambda R)\varphi_1(R) - K_1(\lambda R)\varphi_1'(R)}.$$

It can be shown that the induced change in impedance of the coil due to the presence of a conducting tube is given by the formula

$$\tilde{A}_0^{ind}(r, \lambda) = C_2 I_1(\lambda r), \tag{28}$$

where C_2 is given by (27).

Applying the inverse cosine transform of the form

$$A_0^{ind}(r, z) = \frac{2}{\pi} \int_0^\infty \tilde{A}_0^{ind}(r, \lambda) \cos \lambda z d\lambda$$

to (28) we obtain

$$A_0^{ind}(r, z) = \frac{2}{\pi} \int_0^\infty C_2 I_1(\lambda r) \cos \lambda z d\lambda. \tag{29}$$

The change in impedance of the coil is given by (see [3])

$$Z_{ind} = \frac{j\omega}{I} \oint_L \tilde{A}_0^{ind}(r, z) dl, \tag{30}$$

where L is the contour of the coil. Substituting (29) into (30) we obtain

$$Z_{ind} = -2\omega\mu_0 r_1^2 Z,$$

where

$$Z = jr_0 \int_0^\infty I_1^2(\lambda r_0) \frac{D}{E} d\lambda. \tag{31}$$

The results of numerical computations of the change in impedance Z using formula (31) are shown in Fig.1. The three curves in Fig. 1 correspond to three different values of α, namely, $\alpha = -1/2, -1$ and -2 (from bottom to top). The following values of the other parameters are used for calculations: $\mu_1 = 1,\ r_0 = 0.8,\ R = 1.2,\ \sigma_* = 1,\ \mu_* = 1.$ The points on each curve in Fig. 1 correspond to different values of the parameter $\delta = r_1\sqrt{\omega\sigma\mu_0}$ (δ increases from 2 to 8 from left to right in Fig. 1). Calculations are performed using MATHEMATICA since confluent hypergeometric functions and Bessel functions are the built-in functions in MATHEMATICA.

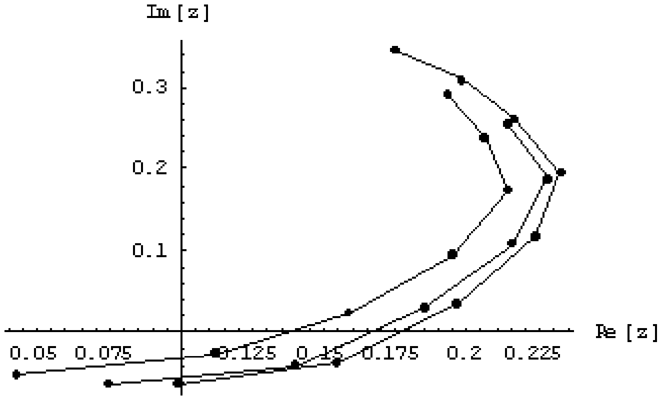

Fig. 1. The change in impedance computed by means of (31) for three values of α

IV. CONCLUSIONS

Analytical solution for the change in impedance of a single-turn coil located inside a conducting tube is obtained in the present paper. It is assumed that the electric conductivity and magnetic permeability of the tube are power functions of the radial coordinate in the form $\mu = \mu_* r^{\alpha}$, $\sigma = \sigma_* r^{\beta}$, where α and β are given constants. A one-parameter family of closed-form solutions for the case $\alpha + \beta = -1$ is presented in the paper in terms of different special functions. The corresponding ordinary differential equation for the Fourier cosine transform of the vector potential in the conducting tube is solved in terms of the confluent hypergeometric function. Results of numerical calculations are presented in the paper. The method of solution can be easily generalized in case of a multilayer tube with varying electric conductivity and magnetic permeability.

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Valentīna Koliškina, Inta Volodko. Analītisks atrisinājums elektromagnētiskās nesagraujošās kontroles problēmai vadošā caurulē ar mainīgām vides īpašībām
Rakstā iegūta impedances izmaiņas aprēķinu formula vijumam ar strāvu, kurš atrodas vadošas bezgalīgi garas cilindriskas caurules iekšpusē. Vijuma ass sakrīt ar caurules asi. Tiek pieņemts, ka caurules parametri (elektriskā vadāmība un magnētiskā caurlaidība) ir radiālās koordinātas pakāpes funkcijas. Vektorpotenciāls rēķināts trīs dažādos apgabalos: gaisā caurules iekšpusē, caurules cilindriskajā slānī un gaisā ārpus caurules. Diferenciālvienādojumu sistēma vektorpotenciāla aprēķināšanai risināta analītiski, pielietojot Furjē kosinusa transformāciju. Parastais diferenciālvienādojums vektorpotenciālam vadošā cilindriskā caurulē ar mainīgām elektriskām un magnētiskām īpašībām risināts, izmantojot deģenerēto hiperģeometrisko funkciju. Atrisinājums vektorpotenciālam gaisā gan caurules iekšpusē, gan ārpusē izteikts ar modificētām Beseļa funkcijām. Atrisinājums inducētam vektorpotenciālam iegūts neīstā integrāļa veidā. Formula vijuma impedances izmaiņas aprēķināšanai iegūta, integrējot inducēto vektorpotenciālu pēc vijuma kontūra. Impedances izmaiņas skaitliskie aprēķini atkarībā no dažādiem problēmas parametriem veikti, izmantojot programmu paketi „Mathematical”. Rakstā iegūto analītisko atrisinājumu vektorpotenciālam var viegli vispārināt vairāku slāņu cilindriskas caurules gadījumam ar dažādu elektrisko vadītspēju un magnētisko caurlaidību.

Валентина Колышкина, Инта Володко. Аналитическое решение задачи электромагнитного неразрушающего контроля для проводящей трубы с переменными свойствами
В статье получена формула для расчета изменения импеданса витка с током, расположенного внутри проводящей цилиндрической трубы. Ось витка совпадает с осью трубы. Предполагается, что параметры цилиндрической трубы (электропроводность и магнитная проницаемость) являются степенными функциями радиальной координаты. Вектор-потенциал рассматривался в трёх областях: в воздухе внутри трубы, в проводящей цилиндрической трубе и в воздухе вне трубы. Система уравнений для вектор-потенциала решена аналитически с помощью метода косинус-преобразования Фурье. Обыкновенное дифференциальное уравнение в проводящей цилиндрической трубе с переменными электрическими и магнитными свойствами решено с использованием вырожденной гипергеометрической функции. Вектор-потенциал в воздухе как внутри трубы, так и вне трубы выражен через модифицированные функции Бесселя. Решение для индуцированного вектор-потенциала получено в виде несобственного интеграла. Формула для расчета изменения импеданса витка получена интегрированием индуцированного вектор-потенциала по контуру витка. Проведены числовые расчеты изменения импеданса в зависимости от параметров задачи с использованием программного пакета „Mathematica”. Полученное в статье аналитическое решение для вектор-потенциала может быть обобщено на случай многослойной цилиндрической трубы различной электропроводности и магнитной проницаемости.