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The Evaluation of Financial Assets with Autocorrelations in Returns

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Abstract **- In this paper, we will describe an analytical solution to a problem of pricing financial assets with autocorrelations in returns. We will develop a continuous diffusion model for the case of autocorrelation in stock returns, obtain the European call option pricing formula written on a stock with autocorrelation in returns and show that even small levels of predictability due to autocorrelation can give a substantial deviation from the results obtained by Black-Sholes formula. Also, we will calculate the modified sensitivities of the value of European call option and show how in risk management widely used option hedging parameters depend on assumptions made about correlation in underlying asset returns. Finally, we will show convergence time for the stationary solution of the derived continuous diffusion model and test its distribution.**

Keywords**: Discrete time stochastic difference equation systems, ARCH models, Markov chain, Black–Scholes model, option Greeks**

I. INTRODUCTION

In recent years, a lot of papers have been dedicated to the problem of autocorrelation in asset returns and its predictability. Autocorrelation in short-term stock index returns has been analyzed by Lo and MacKinlay in [1], Jokivuolle in [2] and Stoll and Whaley in [3]. They argue that positive autocorrelation shows up in index returns due to presence of stale prices of stocks included in the index. The above-mentioned situation happens when the increase in the number of stocks comes from inclusion of small capitalization stocks, which are known to trade less frequently than large ones. Due to infrequent trading in small capitalization stocks, the observed index value does not reflect the true market value of the underlying stock portfolio as the index value is calculated using the last observed stock transaction prices. For such process modelling purpose, econometricians have been very active in developing models of conditional heteroskedasticity. But when your discrete model contains unobservable state variables (like conditional variance) in the system, the likelihood of a nonlinear stochastic equation system observed at discrete intervals can be very difficult to derive. Nelson [4] was one of the first to partially bridge the gap by developing conditions under which ARCH stochastic difference equation systems converge in distribution to Ito processes as the length of the discrete time intervals goes to zero. This solution was performed for the models without autocorrelation in returns. Thus, the problem of autocorellation during the life of the financial asset can broadly change the value of asset and its derivative. Undervalued derivatives as we know from the past can lead to a huge bankruptcy chain of financial institutions.

The paper is structured as follows. Section 2 gives a brief review of the autocorrelation problem in asset returns and the idea of diffusion approximation. Section 3 describes Black–Scholes European Call option price formula and market risk sensitivity modification. In Section 4, we report our test results for the distribution of the stationary solution and convergence time to stationary solution based on fixed parameters, but Section 5 concludes and discusses several possible avenues for future research.

II. THE PROBLEM OF AUTOCORRELATION IN ASSET RETURNS

Mezin [5] found a relationship among asset price volatility, asset return volatility and asset return autocorrelation coefficient. The obtained analytical solution reduces to the well-known Black–Scholes option pricing formula for the special case of autocorrelation in asset returns. Mezin created a framework of a lognormally distributed asset price *S* with serially correlated returns and derived an analytic option pricing model, capable of providing an exact solution for a value of derivatives on such an asset. He developed framework of random, normally distributed, process x, such that $lnS = X$, with autocorrelated increments ξ that have volatility σ^2 and autocorrelation coefficient ρ. Both parameters can be estimated using historical data. Instead of heuristically based approach it is possible to assume limit theorems proposed by Carkovs [6] and get the continuous time approximation of stochastic difference equation in a form of diffusion approximation.

The simplest mathematical model describing development of stock price S_t and involving assumption of autocorrelation in stock returns under commonly used condition on risk neutrality of probabilistic measure P may be written in the following way:

$$S_{t+1} = S_t(1+\varepsilon^2\mu + \varepsilon\sigma y_{t+1}) \quad (1)$$

where y_t is a Gaussian random sequence with zero mean and unit variance. When it is considered that these random numbers are independent, we may write that y_t follows AR(1):

$$y_{t+1} = \rho y_t + \sqrt{1-\rho^2}\,\xi_{t+1} \quad (2)$$

where $\{\xi\}_t$, $E\xi_t = 0$, $E\xi_t^2 = 1$ is i.i.d. Gaussian sequence. To be able use results derived by Carkovs in [6], we denote $x_t \equiv S_t$ and rewrite equation (1) in the following form:

$$x_{t+1} = x_t + \varepsilon\sigma y_{t+1}x_t + \varepsilon^2\mu x_t \quad (3)$$

These results state that for small ε, equation (3) can be approximated by distribution of vector $\{X(t_1), X(t_2), \ldots, X(t_n)\}$ defined by solution of Ito stochastic differential equation:

$$dX(s) = a(X(s))ds + \sigma(X(s))d\omega(s) \qquad (4)$$

Thus, we derive continuous time approximation of stochastic difference equation (1) in a form of diffusion process satisfying Ito stochastic differential equation:

$$dS(t) = S(t)(\mu + \sigma^2 k)dt + S(t)\sqrt{1+2k}\,\sigma d\omega(t) \qquad (5)$$

$$k := \sum_{m=1}^{\infty} Corr\{y_{t+m}, y_t\} = \frac{\rho}{1-\rho}$$

After substitution we get the final equation

$$dS(t) = S(t)(\mu + \sigma^2 \frac{\rho}{1-\rho})dt + S(t)\sqrt{\frac{1+\rho}{1-\rho}}\,\sigma d\omega(t) \qquad (6)$$

III. European Call Option Pricing on Stocks with Autocorrelation in Returns and Market Risk Sensitivity Modifications

Now let us derive European call option pricing formulas if underlying stock price process *S(t)* satisfies the stochastic differential equation (6). The boundary conditions for the European call option are given $C(S(T),T) = \max(S(t) - K; 0)$ and $C(0,t) = 0$. Using well-known techniques, we get the following results

$$C(S(t),t) = S(t)N(d_1) - K\exp(-(\mu+\sigma^2 k)(T-t))N(d_2) \qquad (7)$$

where

$$d_1 = \frac{\log(S(t)/K) + (\mu + \sigma^2 k + \frac{1}{2}\sigma^2(1+2k))(T-t)}{\sigma\sqrt{(1+2k)(T-t)}}$$

and

$$d_2 = d_1 - \sigma\sqrt{(1+2k)(T-t)}$$

where N() is the standard normal cumulative distribution function.

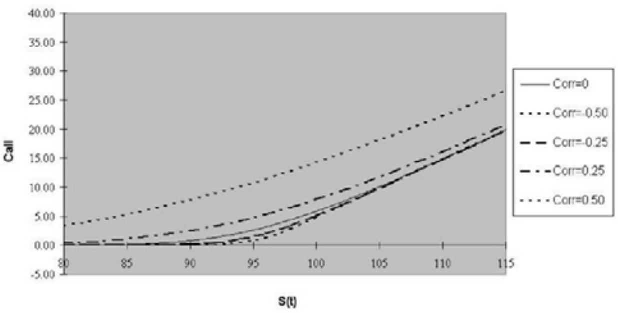

Fig. 1. Value of call option for different correlation coefficients

Now we are ready to derive formulas used to calculate sensitivities of call option price to changes in underlying parameters. There are a lot of risk measures, which are mostly defined with Greek letters, but to illustrate the autocorrelation problem let us take a look at the most common Greek letters used in risk measurement – Delta and Gamma.

Delta, Δ, is the first derivative of the value C (European call option price) of the option with respect to the underlying instrument price S:

$$\Delta(S(t),t) = \frac{\partial C}{\partial S} = N(d_1)$$

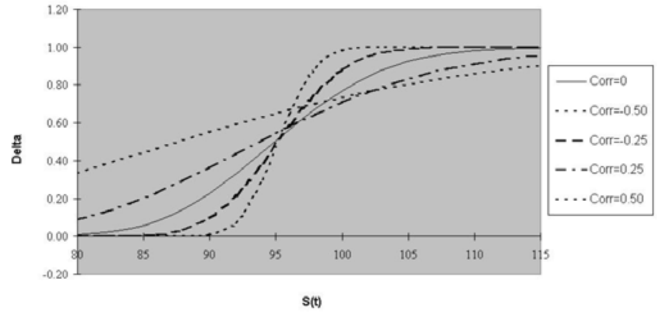

Fig. 2. Value of call option delta for different correlation coefficients

Gamma, Γ, measures the rate of change in the delta with respect to changes in the underlying asset price:

$$\gamma(S(t),t) = \frac{\partial \Delta}{\partial S} = \frac{N(d_1)}{S(t)\sigma\sqrt{(1+2k)(T-t)}}$$

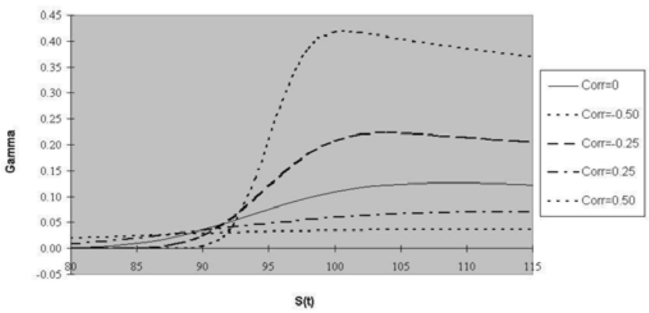

Fig. 3. Value of call option gamma for different correlation coefficients

As shown in Fig. 2 and 3, option price sensitivities depend on the autocorrelation in underlying asset returns. If there is autocorrelation, then depending on its sign, an option seller can overweight or underweight a market risk, and as a result this could lead to unpredictable losses or even to institution's default.

IV. Testing Distribution of Stationary Solution and Finding Convergence Time Based on Fixed Parameters

The next step, which we cannot exclude from our research, is stability of the solution of the equation (4) and the derived equation (5). If we cannot find a stationary solution to the equation (5) and show that it is independent on the level of the correlation coefficient then the above-mentioned formulas for the European call option price and Greeks are not sustainable

and describe only a local timely solution, which is useless in risk management.
Let us take into consideration the stochastic differential equation, which is similar to equation (5):

$$dx(t) = \gamma(\rho) - a(\rho)xdt + \sigma(\rho)xd\omega(t) \quad (8)$$

$$\hat{x}(t) - stationary$$

We know the distribution of stationary solution $\hat{x}(t)$ from Nelson [4] research (it is inverse Gamma distribution). If we intend to find the speed of convergence to the stationary solution for equation (8) with an initial condition $x(0) = x$ it should be rewritten in the different form:

$$z(t) = x(t) - \hat{x}(t) \qquad z(0) = x - \hat{x}(0)$$

$$dz(t) = a(\rho)z(t)dt + \sigma(\rho)z(t)dw(t)$$

$$E\{|z(t)|\}^2 = E\{|z(0)|^2\}e^{\lambda_2(\rho)t}$$

$$E\{|z(0)|^2\} = \int_0^\infty p(z)(x-z)^2dz$$

where $p(z)$ is Γ- distribution.

And if $\lambda_2(\rho) = 2a(\rho) + \sigma^2(\rho) < 0$, then the second order moment is super martingale

$$\text{c}(\varepsilon) := \text{P}\left\{\sup_{T \le t} |z(t)|^2 > \varepsilon\right\} \le$$

$$\le \frac{1}{\varepsilon}\text{E}\left\{\sup_{T \le t} |z(t)|^2\right\} \le \frac{4}{\varepsilon}\text{E}\{|z(T)|^2\} =$$

$$= \frac{4}{\varepsilon}\text{E}\{|z(0)|^2\}e^{\lambda_2(\rho)T}$$

Second order moment $E\{|z(0)|^2\}$ with known Γ-distribution (this distribution depends on ρ) is easily evaluable. If the inequality mentioned below were in force

$$\text{P}\left\{\sup_{T \le t} |z(t)|^2 < \varepsilon\right\} > 1 - \delta$$

it should take convergence time:

$$T_\varepsilon(\rho) = \frac{\ln(\delta\varepsilon) - \ln 4\left(\text{E}\{|z(0)|^2\}\right)}{\lambda_2(\rho)} \quad (9)$$

Formula (9) describes convergence time to stationary solution $\hat{x}(t)$. Thus, using (9) depending on ρ it is possible to find convergence time, and from that moment of time stationary solution should have Γ- distribution. In our studies we have fixed parameters in (8) and Γ- distribution. After that we have made simulations by changing ρ values from 0 to 1 and as a result we obtained different convergence time. Taking into account Nelson [4] results of stationary solutions, we have made Kolmogorov tests for Γ- distribution. From our point of view, these tests should be empirical proving (9). For most cases, we have not rejected the hypothesis of the Γ-distribution.

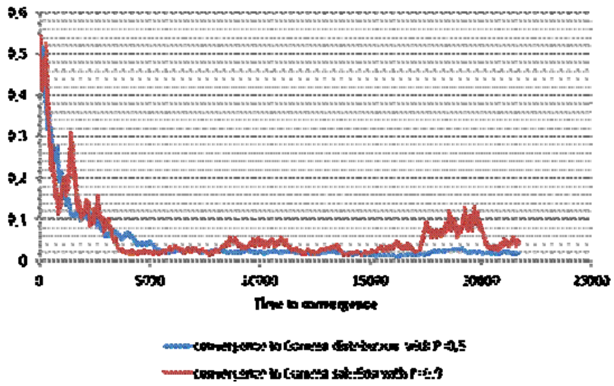

Fig. 4. Convergence time to gamma distribution for different correlation coefficients

As we see from Fig. 4, for given precise $\varepsilon < 0{,}1$, $|z(t)|^2$ converge faster to the stationary solution with correlation coefficient 0.5 than 0.9. However, convergences have been achieved for all ρ values. It means that equation (8) can be used for the price evaluation of financial assets with autocorrelation in returns and the determination of risk measures.

V. Conclusions and Further Work

We have developed a continuous diffusion model for the case of autocorellation in stock returns. Further, we have obtained the European call option pricing formula written on the stock with autocorrelation in returns and shown that even small levels of predictability due to autocorrelation can give a substantial deviation from the results obtained by Black–Sholes formula. Also, we have derived formulas for sensitivities of the value of European call option and shown how in risk management the widely used option hedging parameters depend on assumptions made about correlation in underlying asset returns. The approach can be applied to discrete time stochastic difference equation systems, where volatility is stochastic or it is modelled by generalized autoregressive conditional heteroscedasticity process.

Another uncovered topic is how the convergence of discrete time stochastic difference equation to its continuous time approximation depends on the autocorrelation coefficient.

References

[1] Lo, A., MacKinlay, A. C.: An Econometric Analysis of Non-Synchronous Trading. Journal of Econometrics, No. 45, pp. 181-211, 1990.

[2] Jokivuolle, E.: Measuring True Stock Index Value in the Presence of Infrequent Trading. Journal of Financial and Quantitative Analysis, September, 1995.
[3] Stoll, H. R., Whaley, R. E.: The Dynamics of Stock Index and Stock Index Futures Returns. Journal of Financial and Quantitative Analysis, No. 25, pp. 441-68, 1990.
[4] Nelson, D. B.: ARCH models as diffusion approximations. Journal of Econometrics, Vol. 45, Issues 1-2, Pages 7-38, July-August 1990.
[5] Mezin, V. Option Pricing Model for Autocorrelated Stock Returns. Working paper, Rutgers University, 2004.
[6] Carkovs,J. On Diffusion Approximation of Discrete Markov Dynamical Systems. In Computational Geometry, Proceedings of World Academy of Science, Engineering and Technology, Vol. 30, pp.1-6, 2008.

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Aigars Egle, Andrejs Matvejevs, Jegors Fjodorovs. Finanšu aktīvu novērtēšana ar autokorelēto ienesīgumu
Šajā rakstā tiek aprakstīts analītiskais risinājums finanšu aktīvu cenu noteikšanas problēmai ar autokorelēto ienesīgumu. Par darba pamatu kalpo dažādu zinātnieku pētījumu rezultāti [1], [2] un [3], kas tika veltīti akciju indeksa autokorelācijai, kad šajā indeksā tika iekļautas mazās kapitalizācijas kompānijas. Rezultāti parāda, ka eksistē pozitīva korelācija, kad ir iekļauts indeksā liels mazlikvīdo akciju skaits. Dažreiz indeksa patiesā cena ir deformēta tāpēc, ka akciju cenas bieži nemainās. Tādējādi, lai modelētu tādas akciju tirgus parādības, ekonometristi attīstījuši ARCH tipa modeļus, kuros nosacītā dispersija varētu mainīties laikā. Bet, kad diskrētais modelis satur nenovērojamo mainīgo (piem., nosacīto dispersiju), varētu būt grūti atrast ticamības funkciju nelineārai stohastisko vienādojumu sistēmai, definētai diskrētos intervālos. Nelsons [4] bija pirmais, kas attīstīja nosacījumus, pie kuriem diferenču tipa ARCH modelis konverģē pēc varbūtības uz Ito procesu, kad diskrēto laika intervālu garums samazinās līdz nullei. Savukārt visa iepriekšminētā ideja tika izstrādāta gadījumā, ja aktīvu ienesīgums nav autokorelēts, bet reālajā situācijā šī korelācija pastāv, kas būtiski varētu izmainīt novērtējumu. Līdz ar to šajā darbā tiek izstrādāts nepārtraukts difūzijas modelis (kura pamatā ir J.Carkova darbi par diferenču vienādoju aproksimāciju ar stohastisko diferenciālvienādojumu skat. [6]) gadījumā, kad akciju ienesīgums ir sērijveidā autokorelēts, iegūts Eiropas tipa opcijas cenu noteikšanas vienādojums, kurš ņem vērā autokorelāciju, kā arī tiek parādīts, ka nelielas sērijveida autokorelācijas līmeņa izmaiņas varētu novirzīt opcijas cenu no rezultāta, iegūta ar Black-Sholse formulas palīdzību. Šajā darbā tiek izskaitļotas Eiropas tipa opcijas cenu jutīguma izmaiņas un parādīts, ka opciju cenu noteikšanas risku vadībā plaši izmantotie hedžēšanas parametri (Greeks) ir atkarīgi no aktīvu korelācijas. Tādējādi, izmantojot plaši pazīstamos opciju cenu svārstības riska ierobežojošos rādītājus – *Greeks,* varētu būt gadījumi, kad opciju izrakstītājs nenovērtā iespējamās izmaiņas bāzes aktīvā, kā rezultātā šī nepareizā rīcība varētu novērst pie zaudējumiem vai pat līdz finanšu institūcijas bankrotam. Bez tam brīvi izvēlētam nepārtrauktam difūzijas modelim atrisinājuma konverģences laiks tiek noteikts uz stacionāru un izveidoti statistiskie testi uz stacionārā atrisinājuma sadalījumu.

Айгарс Эгле, Андрей Матвеев, Егор Фёдоров. Оценка финансовых активов с автокоррелированными доходностями
В этой статье описаны исследования для аналитического решения проблемы ценообразования финансовых активов с автокоррелированными доходностями. В основе работы лежат труды различного периода о поведении финансовых рынков [1], [2] и [3], которые в основном посвящены автокорреляции индекса акций. Результаты трудов показывают, что существует позитивная автокорреляция, когда в индекс включено большое количество малоликвидных акций. Таким образом, настоящая цена индекса деформирована, потому что цены акций редко меняются. Поэтому, для моделирования таких процессов эконометристы используют ARCH модели, в которых условная дисперсия может меняться во времени, но когда дискретная модель содержит ненаблюдаемую переменную, могут появиться сложности с оценкой параметров нелинейной модели. Нельсон [4] был первый, кто предложил условия сходимости по вероятности ARCH модели на процесс Ито, когда интервал дискретного времени стремится к нулю. Вся вышеупомянутая теория разработана на случай отсутствия автокорреляции, что не всегда согласуется с реальностью. Мы разработали непрерывные модели диффузии для случая серийной корреляции доходностей акций (более подробно с принципами аппроксимации и её сходимости можно ознакомиться в работе Е.Царькова [6]), получили уравнение для цены европейского опциона, которое учитывает автокорреляцию, а также показали, что даже небольшие уровни серийной автокорреляции могут дать существенное отклонение от результатов, полученных при помощи формулы Блэка- Шоулса. В работе вычислено изменение чувствительности стоимости европейского опциона *call* и показано, что в области управления рисками ценообразования опционов, широко используемые параметры допустимости хеджирования *(Greeks)* зависят от предположений о корреляции, лежащих в основе доходности активов. Таким образом, используя показатели рыночного риска опционов (так называемые – *Greeks)* как основу управления позициями в портфеле активов либо обязательств, для устранения колебаний цены опциона от изменения рыночной конъюнктуры основного актива, необходимо учесть автокорреляцию доходностей, что уменьшает вероятность потерь при продаже опционов. Наконец, мы показали время сходимости для стационарного решения производной непрерывной диффузионной модели и сделали статистические тесты на его распределение.