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Some Approaches to Combining Probabilistic and Fuzzy Uncertainties

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***Abstract* – Different types of uncertainty are widely spread in all areas of human activity. Probabilistic uncertainties are related to the chances of occurrence of random events. To deal with this kind of uncertainty, statistics and probability theory are successfully employed. Another kind of uncertainty, fuzzy uncertainties refer to imprecision and fuzziness of different kinds of measurements. To cope with this kind of uncertainty, the fuzzy set theory is used. This paper addresses widespread approaches to combining probabilistic and fuzzy uncertainties. The theoretical fundamentals of the approaches are considered within the context of the generalized theory of uncertainty (GTU).**

***Keywords* – probabilistic uncertainty, fuzzy uncertainty, random variable, generalized constraint, fuzzy event, probability of fuzzy event, fuzzy probability distribution**

I. Introduction

Uncertainties accompany practically all kinds of human activities. What is the partner's hand in a card game? What will be the profitability of company shares in a year? What is the distance to the object when it is measured by eye? The list of uncertainties like that can be easily extended.

How can we take into consideration, evaluate and analyse different kinds of uncertainties? Historically, the first type of uncertainty to deal with which the theoretical and practical foundations were elaborated, were the chances of random events occurrence. A push to the development of probability theory was the need to evaluate the chances of players in different hazardous games. Probability theory has gone a long way of development. Nowadays it provides a sound mathematical apparatus for dealing with specific uncertainties in different areas of human activity. Hereinafter, we will call such uncertainties probabilistic.

Statistics can provide ways to practically evaluate the parameters of different kinds of probability distributions and stochastic dependencies. But statistics and probability theory cannot help to represent vague subjective opinions about the distance to an object. Here we face another kind of uncertainty: ambiguity, imprecision or fuzziness. To cope with this kind of uncertainty, fuzzy set theory can be applied successfully. Further we will call that kind of uncertainty *fuzzy uncertainties*.

Let us extend consideration frames. Let us assume that the range of relevant variable changes is divided into fuzzy subsets. How can we evaluate the probability that a real value of that variable will belong to a certain fuzzy subset? Let us consider another problem. Let the function of probability distribution of some random variable be constructed. Due to different disturbing factors, it is impossible to construct a precise distribution. For each specific value of the random variable, only a fuzzy subset can be determined, to which the value of distribution function can belong. How should we then construct such a fuzzy distribution function?

Neither probability theory, nor fuzzy set theory can provide any answers to the above-mentioned questions. The answers requested can only be obtained through combining the probabilistic and fuzzy approach. This paper examines some widespread approaches to this kind of combination. The theoretical foundations of the approaches are discussed within the generalized theory of uncertainty (GTU).

II. Theoretical Foundations of Combination of Uncertainties

Nowadays a lot of publications are available offering different methods for combining probabilistic and fuzzy uncertainties. Most of these techniques deal with solving specific tasks. On the other hand, the knowledge and experience acquired in the field of analysis and management of uncertainties of different kind have initiated attempts to develop a generalized theory of uncertainty that would include separate kinds of uncertainty as special cases. One of the most successful works in that field is the generalized theory of uncertainty (GTU), whose detailed fundamentals are given in [9]. As the author shows, GTU differs three-fold from other approaches to managing uncertainties. First, the statement that information is statistical in nature is replaced by a statement that information is a generalized constraint. The concept of a generalized constraint plays a central role in GTU. Due to that, the concepts of graduation and granulation are introduced. The concept of graduation refers to the aspect that any relevant values can be related to the degree of membership in certain groups of values (in particular, in fuzzy sets). It is assumed that relevant values are, or are allowed to be granulated. A granule is a group of values drawn together by indistinguishability, similarity, proximity or functionality [9]. In general case, the concepts of graduation and granulation can be related to the concept of a linguistic variable. Any linguistic variable includes a fuzzy set of values of relevant variable, as well as a set of variable membership values. Second, the theory abandons using the concept of bivalence and employs fuzzy logic. Third, the theory implies using techniques that enable handling information described in the natural language.

To ensure that information expressed in the natural language can be processed by formal methods, it has to be transformed into the form enabling the direct use of respective techniques. To transform the initial information into a suitable formal representation, the GTU uses the concept of precisiation. As mentioned in [9], precisiation and precision have different facets. More specifically, it is recommended considering precisiation as λ-precisiation, where λ is an index variable, whose modalities identify various modalities of precisiation.

Relevant meanings obtained as a result of precisiation can be divided into two classes [9]: (1) i-meaning that is in essence an intension of the initial information; i-meaning is related to certain numerical evaluations of the initial information, e.g., attribute values in the task of object classification, and (2) extension or e-meaning. This kind of meaning is used to describe entities, when they are not characterized by numerical values. It is clear that in general case i-meanings are more informative than e-meanings.

Some generalized constraints that bear a direct relation to the topic of this study are considered below.

A. Probabilistic Constraints

In a standard case, such constraints can be set in different forms, e.g., in the form of probability distribution function or continuous random variable probability density function, in the form of a list of random events and corresponding to them probabilities. In GTU, a generalized probabilistic constraint can also be defined in fuzzy subsets. A simple example illustrating the concept is shown in Fig. 1.

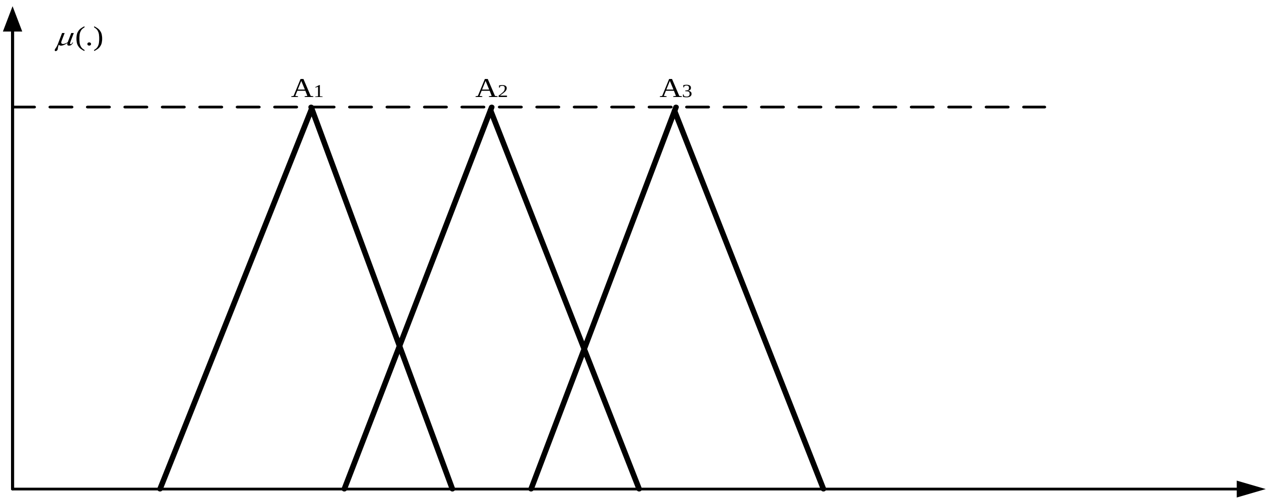

Fig. 1. Illustration of the concept of a generalized probabilistic constraint in GTU

Let a set of objects be evaluated by means of attribute A. Since it is impossible to determine precise values of the attribute for the object, it is divided into three fuzzy subsets: using previously defined linguistic variables A_1, A_2 and A_3. Then a fuzzy event will be falling of the real value of the attribute to one of fuzzy subsets. From Fig. 1 it follows directly that a real value of the attribute can at the same time belong to two fuzzy subsets. In this case, we have to speak about simultaneous implementation of two fuzzy events.

Formally, the probability of fuzzy event A can be defined as [8]

$$\Pr(A) = \int_{-\infty}^{\infty} \mu_A(x) f(x) dx = E\left(\mu_A(x)\right), \tag{1}$$

where f(x) – the probability density function for a relevant random variable.

If a fuzzy event is determined by a fuzzy subset in the interval [a, b] of relevant random variable, then

$$\Pr(A) = \int_{a}^{b} \mu_A(x) f(x) dx = E\left(\mu_A(x)\right). \tag{2}$$

For a discrete sample with consequences $x_1, x_2, \ldots, x_n$ expression (2) will look as follows:

$$\Pr(A) = \sum_{i=1}^{n} \mu_A\left(x_i\right) f\left(x_i\right) = E\left(\mu_A(x)\right). \tag{3}$$

In any case, the probability value of fuzzy event A is equal to the mathematical expectation of relevant values of the membership function.

If fuzzy event A is made by implementation of random variable $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$, , respectively, the entropy of fuzzy event A can be calculated as [7]:

$$H^P(A) = -\sum_{i=1}^{n} \mu_A\left(x_i\right) p_i \log p_i. \tag{4}$$

B. Usuality Constraints

The construction of probability distributions and evaluation of their parameters are performed on the basis of statistical data (samples). These factors essentially affect the process of statistical data processing [3]:

1. Uncertainty due to the correlation between the data being observed and universe of possible data.

2. Imprecision of empirical phenomenon measurement, which leads to an increase in statistical data imprecision.

3. Vagueness related to the use of linguistic terms in the real world description.

4. Partial or complete non-knowledge connected to the value of phenomenon in the examples under observation.

5. Imprecision caused by granulation of terms used in the description.

If due to some reasons it is impossible to unambiguously evaluate implementation of a random variable, it can be evaluated as fuzzy subsets. In this case we deal with a fuzzy random variable (FRV). Conditional fuzzy variable implementation is shown graphically in Fig. 2.

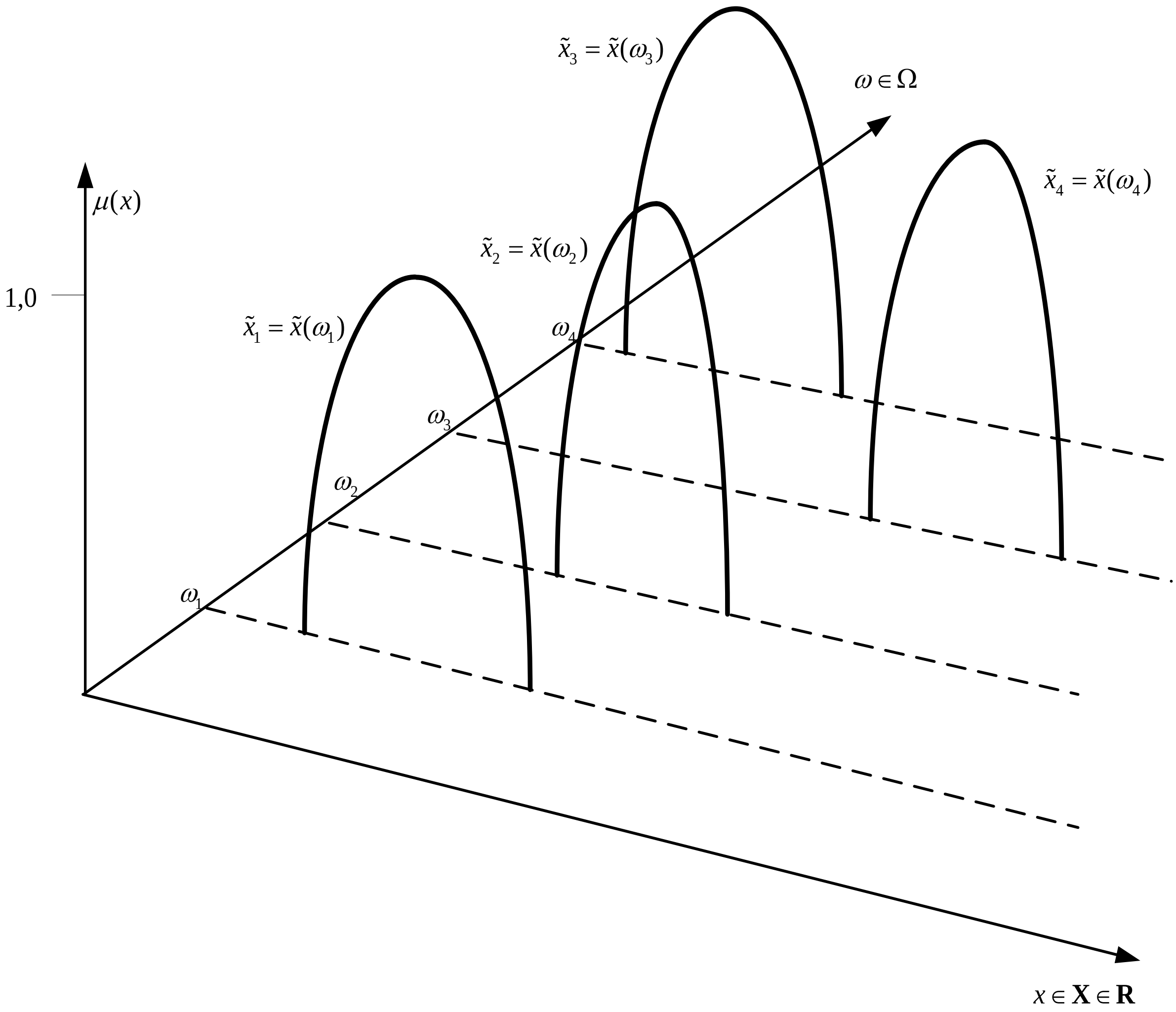

Fig. 2. Graphical representation of implementation of a sample fuzzy random variable

There are several approaches to formal definition of a FRV. A technique discussed in [5] is the most common one; according to it, probabilistic space (Ω, A, P) is a statistical mechanism generating a FRV. Then a FRV is determined by means of mapping

$$X : \Omega \rightarrow F_C(\mathbf{R}),$$

such that for each value $\alpha \in [0, 1]$ the mapping of α-level

$$X_\alpha : \Omega \rightarrow K_C(\mathbf{R}) \text{ with}$$

$$x_\alpha(\omega) = \left[\inf\left(x(\omega_\alpha)\right), \sup\left(x(\omega_\alpha)\right)\right], \forall \omega \in \Omega$$

is a compact convex random set, i.e., Borel-measurable in the Borel σ-field, generated by means of topology related to Hausdorff metric. The assumption of Borel-measurability enables handling generated random sets by analogy with probabilities and constructing probabilistic distributions of FRV.

As an illustration, Fig. 3 depicts the sample graphs of distribution function and distribution density function of a sample fuzzy random variable.

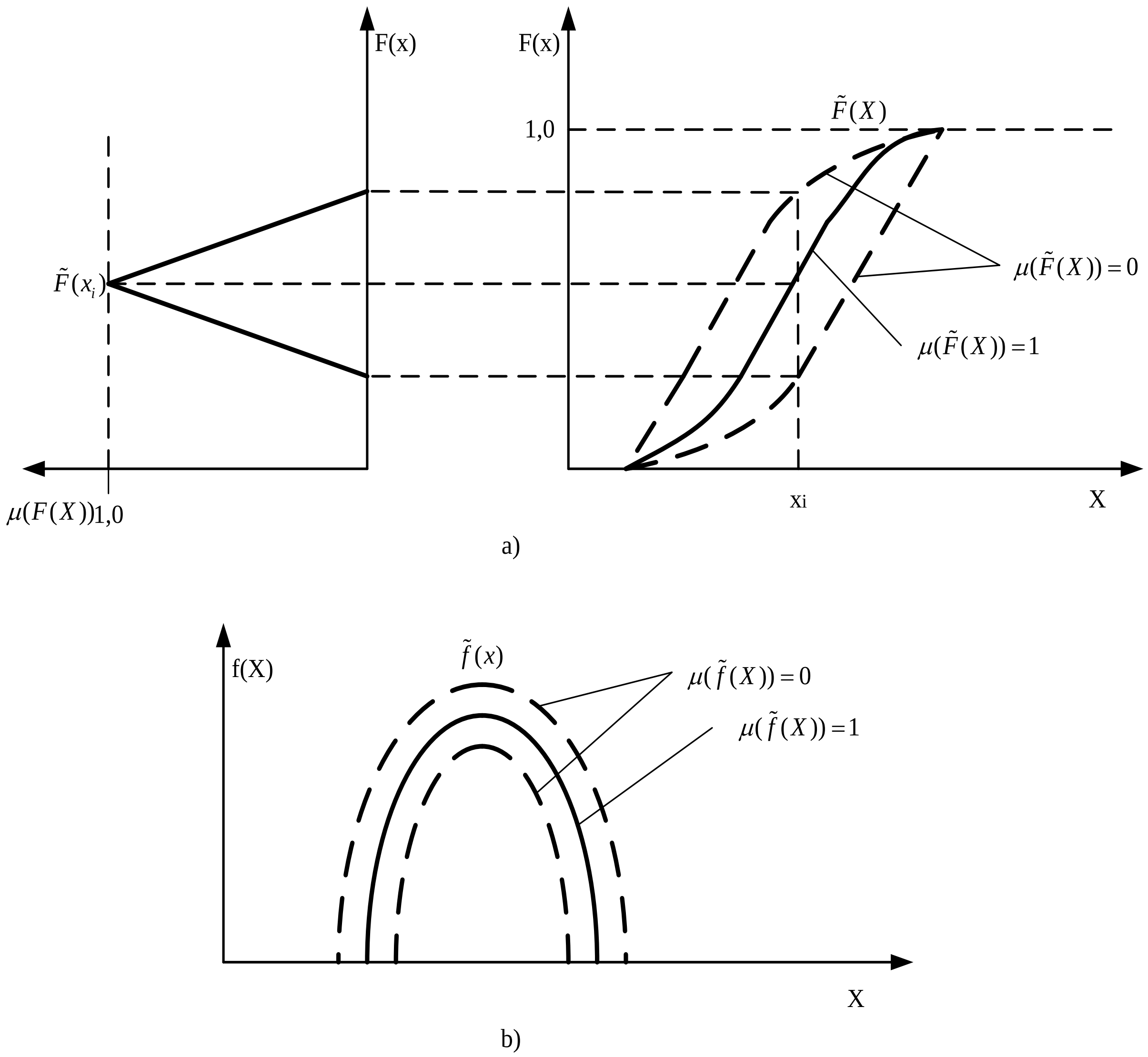

Fig. 3. (a) a sample graph of distribution function and (b) a sample graph of distribution density function of a sample fuzzy random variable

As can be seen from Fig. 3, for the given value of a random variable x_i, the value of distribution function is a fuzzy triangular number and any value of distribution function is related to the corresponding membership value.

III. Some Examples of Practical Application of Uncertainty Combination Methods

Let us start this section with discussing some examples of practical use of probabilities of fuzzy events. In [4], a specific method for constructing fuzzy classifiers is proposed. The essence of fuzzy classifiers is as follows. Assume that a training set of instances is specified. For each instance, its membership in one of classes, $c_k \in C$ is known. Each instance is evaluated using a set of attributes A. For every attribute $A_j \in A$, a set of its fuzzy values $a_{ji} \in A_j$ is defined. As a measure of attribute informativity, the notion of the so-called star entropy is used:

$$H_s^*(C/A_i) = -\sum_{j=1}^{m_j} p^*(a_{ij}) \sum_{k=1}^{n} p^*(c_k/a_{ij}) \log_2 p^*(c_k/a_{ij}), \tag{5}$$

where $p^*(a_{ij})$ – the probability of a fuzzy event: an instance of the training set has the value of attribute a_{ij} with non-zero value of membership function, $p^*(c_k/a_{ij})$ – the probability of a fuzzy event: an instance with attribute value a_{ij} belongs to class c_k.

At the first step of algorithm execution, the value of star entropy, $H_s^*(C/A_i)$ is calculated using expression (5) for each attribute $A_i \in A$. The attribute that has the largest value of $H_s^*(C/A_i)$ is used as the base one. This attribute corresponds to the root node of a fuzzy decision tree. The branches coming from the root node correspond to the values a_{ij} of attribute A_i. At the end of each branch a training set is represented, which is distinguished using these attribute values. For each of the distinguished subsets, the searching activity of the most informative attribute is performed. A node representing such an attribute is joined to the corresponding branch of a fuzzy decision tree. The process of consecutive construction of a fuzzy decision tree (classifier) is continued until one of stopping criteria is satisfied in all end positions of the tree. Using the constructed classifier, classification of new instances which do not belong to the initial training set is performed.

Another variant of constructing fuzzy classifiers on the basis of the notion of fuzzy statistical entropy is given in [2]. As opposed to the definition of star entropy (5), the fuzzy statistical entropy is defined as follows:

$$H_{sf} = -\sum_{j=1}^{n} \Pr(A_j) \log_2 \Pr(A_j) = = \sum_{j=1}^{n} E(\mu_{A_j}(x)) \log_2 (E(\mu_{A_j}(x))). \quad (6)$$

However, the above-mentioned definition requires that for each value x the sum of values of functions of membership in different fuzzy subsets is equal to 1. Such a requirement is quite restrictive for wide use of the suggested technique.

Let us now consider some examples of the use of fuzzy statistics. Common statistical models enable one to infer knowledge from relevant initial data. Here it is assumed that both initial and inferred data are represented in the deterministic form. It is clear that due to different reasons the inferred data can be more or less uncertain. However, standard statistics has developed plenty of successful methods for modelling this kind of uncertainties.

Fuzzy statistics is designed for operating FRV. Likewise standard statistics, it has multiple directions. Let us consider some of them.

Fuzzy Hypothesis Testing. Hypothesis testing is one of the key techniques in standard statistics. It includes a sequence of procedures that are used to confirm the truth of the proposed hypothesis under the specified conditions and to accept it as a legitimate statement or to show that the hypothesis is false and reject it. Normally, in statistics the null hypothesis is a statement made regarding a certain parameter of distribution, θ, though other statements may be posed as well. An alternative (supplementary or duplicative) hypothesis is a statement that is confirmed when the null hypothesis proves to be false.

Let us consider two conditional hypotheses:

$$H_0 : \theta \in \Psi; \quad H_1 : \theta \in \Theta \setminus \Psi,$$

where θ – a parameter to be analysed;

Ψ – the range of values of parameter θ that is of interest;

$\Theta \setminus \Psi$ – the supplement to the range of all possible values of parameter θ.

The above-mentioned hypotheses can be reformulated in a fuzzy environment as follows:

$$H_0 : \mu_{\Psi}(\theta); \quad H_1 : \mu_{\Theta \setminus \Psi}(\theta).$$

Special approaches are elaborated to make fuzzy testing of the hypotheses. More detailed information about one of such techniques can be found in [1].

Fuzzy Regression Analysis. The main objective of standard regression analysis is to define the stochastic correlation between the values of explanatory variables X_1, …, X_n and the value of response variable Y. In the case of linear regression model, this correlation can be expressed as follows:

$$y_i = b_0 + b_1 x_{i1} + ... + b_n x_{in} + e_i,$$

where b_i, i = 0, …, n, are linear regression coefficients;

e_i – the residual that represents random errors. Commonly it is assumed that $e_i \sim N(0, \sigma^2)$.

In the fuzzy regression model it is assumed that the correlation itself between the explanatory variables and the response variable is fuzzy [6]. According to this model, regression coefficients are fuzzy numbers:

$$\tilde{y}_i = \tilde{b}_0 \oplus \tilde{b}_1 \bullet x_{i1} \oplus ... \oplus \tilde{b}_n \bullet x_{in}, \quad (7)$$

where $\oplus$ – the symbol of fuzzy numbers addition;

$\bullet$ – the symbol of multiplication of a fuzzy number by a real number.

The evaluation of fuzzy regression coefficients $\tilde{b}_{ij}$ is made on the basis of the principle of minimization of the fuzziness extent of response variable $\tilde{y}$ and can be viewed as a specific optimization task. To solve this task, mathematical programming can be employed.

Fuzzy Time Series. If some parameter of a process varies over time, the values of the parameter at different time moments compose a time series. If due to some reasons it is impossible to unambiguously evaluate the values of parameters in time series, then fuzzy values of that parameter are used. In this case one can speak about a fuzzy time series. Using the appropriate techniques, different tasks can be solved on the basis of such data as, for example, forecasting possible values of the relevant parameter at the time points that will be of interest to us in the future.

IV. Conclusions

During the past decades several effective approaches to combining random and fuzzy uncertainties have been developed. This paper has briefly examined two major directions of the approach: probabilities of fuzzy events and fuzzy probabilities. The theoretical foundations of the approaches are described using the generalized theory of uncertainty of L. Zadeh. Besides, some practical applications of the techniques considered are discussed within each area. Based on the analysis of the material provided, it can be concluded that numerous theoretical and practical studies conducted give evidence of the high potential of the suggested techniques for solving complicated practical tasks. However, ever more profound research is required to ensure the development of new techniques and algorithms that will have a strong theoretical validation and at the same time will be simple and easy to use, which is especially topical for the fuzzy regression analysis and fuzzy time series analysis.

REFERENCES

[1] B. F. Arnold and O. Gerke, "Testing fuzzy linear hypotheses in linear regression models", in *Metrika*, 2003, pp. 81-95.
[2] J. vd Berg and U. Kaymak, "On the Notion of Statistical Fuzzy Entropy" in *Soft Methodology and Random Information Systems, Advances in Soft Computing*. Physica Verlag, Heidelberg, 2004, pp. 535-542.
[3] R. Coppi, M. A. Gil and H. A. L. Kiers, "The fuzzy approach to statistical analysis" in *Computational Statistics & Data Analysis (In press)*, www.sciencedirect.com.
[4] Ch. Marsala and B. Bouchon-Meunier, "Choice of a Method for the Construction of Fuzzy Decision Trees", in *Proc. of the 12th International Conference on Fuzzy Systems (FUZZ'03)*, 25-28 May, 2003, Vol.2, pp.584-589.
[5] M. L. Puri and D. A. Ralesku, "Fuzzy random variables" in *J. Math.Anal.Appl*., 1986, 114, pp. 409-422.
[6] H. Tanaka, S. Ueijma and K. Asai, "Linear regression analysis with fuzzy model", in *IEEE Trans. Systems Man Cybernet*., 12, 1982, pp.903-907.
[7] L. A. Zadeh, "Fuzzy sets", in *Information and Control*, 1965, 8, pp.338-353.
[8] L. A. Zadeh, "Probability measures of fuzzy events", in *J. Math. Anal. Appl.*, 1968, 23, pp.421-427.
[9] L. A. Zadeh, "Generalized theory of uncertainty (GTU) –principal concepts and ideas", in *Computational Statistics & Data Analysis*, 2006, 51, pp. 15-46.

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Oļegs Užga-Rebrovs, Gaļina Kuļešova. Dažas pieejas varbūtiskās un izplūdušās nenoteiktības kombinēšanai
Dažāda veida nenoteiktības plaši izplatītas dažādās cilvēku darbības jomās. Vēsturiski pirmais nenoteiktības tips, kuram bija izstrādāti teorētiskie un praktiskie pamati, ir gadījuma notikumu iestāšanās izredzes (varbūtējās nenoteiktības). Varbūtības teorija izgājusi garu attīstības ceļu. Pašlaik varbūtības teorija nodrošina jaudīgu matemātisko aparātu specifisku nenoteiktību vadīšanai dažādās cilvēku darbības jomās. Taču varbūtības teorija un statistika nevar palīdzēt atspoguļot izplūdušus subjektīvus priekšstatus par relevanto reālo lielumu nenoteiktību (izplūdušās nenoteiktības). Lai tiktu galā ar tāda veida nenoteiktībām, veiksmīgi izmanto izplūdušo kopu teoriju. Taču nozīmīga interese ir varbūtējo un izplūdušo nenoteiktību kombinēšanai. Šajā rakstā tiek apskatītas atsevišķas pieejas tādām kombinācijām. Materiāla izklāsts notiek L. Zade vispārīgās nenoteiktību teorijas ietvaros. Šajā teorijā apgalvojuma vietā, ka informācija ir statistiska pēc savas būtības, tiek izmantots apgalvojums, ka informācija ir vispārināts ierobežojums. Otrkārt, teorija neizmanto bivalences jēdzienu, bet izmanto izplūdušo loģiku. Treškārt, šī teorija paredz tādu metožu izmantošanu, ar kuru palīdzību var apstrādāt informāciju, kura tiek izteikta dabīgā valodā. Šajā rakstā tiek minēti atsevišķie izplatītie ierobežojumi, kas atbilst dažādiem nenoteiktību kombinēšanas veidiem. Tiek piedāvāti ilustratīvie izplūdušās sadalījuma funkcijas un izplūdušās sadalījuma blīvuma funkcijas piemēri nosacītam gadījuma mainīgajam lielumam. Tiek apskatīts ilustratīvs izplūdušā gadījuma mainīgā piemērs. Tiek analizēti nenoteiktību kombinēšanas praktiskie piemēri: izplūdušo klasifikatoru konstruēšana, izplūdušo hipotēžu pārbaude, izplūdusī regresijas analīze un izplūdušās laika rindas. Darba nobeigumā ir parādīta turpmāko pētījumu nepieciešamība.

Олег Ужга-Ребров, Галина Кулешова. Некоторые подходы к комбинированию вероятностной и нечеткой неопределенности
Различного рода неопределённости широко распространены в разных областях человеческой деятельности. Исторически первым типом неопределённости, для которого были разработаны теоретические и практические основы, были шансы наступления случайных событий (вероятностные неопределённости). Теория вероятностей прошла большой путь развития. В настоящее время она обеспечивает мощный математический аппарат для управления неопределённости в различных областях человеческой деятельности. Однако теория вероятностей и статистика не могут помочь отобразить смутные субъективные представления относительно действительных значений релевантных неопределённых величин (нечёткая неопределённость). Для обращения с такого рода неопределённостями успешно используется теория нечётких множеств. Однако значительный практический интерес представляет комбинирование вероятностных и нечётких неопределённостей. Данная статья рассматривает некоторые наиболее распространённые подходы к такому комбинированию. Изложение материала производится в рамках общей теории неопределённостей Л. Заде. Согласно этой теории утверждение о том, что информация является статистической по своей природе, заменяется утверждением о том, что информация является обобщённым ограничением. Во-вторых, теория отказывается от использования понятия бивалентности и использует нечёткую логику. В-третьих, эта теория подразумевает использование таких методов, с помощью которых можно обрабатывать информацию, выраженную на естественном языке. В статье представлены некоторые распространённые ограничения, которые определяют различные виды комбинирования неопределённостей. Приведены иллюстративные примеры конструирования нечёткой функции распределения и нечёткой функции плотности распределения условной случайной переменной. Рассмотрен иллюстративный пример реализации нечёткой случайной переменной. Анализируются практические примеры комбинирования неопределённостей: конструирование нечётких классификаторов, нечёткая проверка гипотез, нечёткий регрессионный анализ и нечёткие временные ряды. В заключение показана необходимость дальнейших исследований в данной области.