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Revaluation of Estimated Option Prices Using GARCH Processes with Most Preferable Properties

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***Abstract* - In this paper we describe practical example of our studies for an analytical solution for a problem of pricing financial actives with autocorrelated returns. This example is based on option price for Tesla Motors Inc stock. In previous studies we developed a continuous diffusion model for the case of serially correlated stock returns, obtain European call option pricing formula and show that even small levels of predictability due to serial correlation can give substantial deviation from results obtained by Black-Sholes formula and made Monte Carlo simulation for stock volatility. Finally, we showed recalculated option price, based on the simulation result.**

***Keywords* - Discrete time stochastic difference equation systems, GARCH models, Markov chain, Black Scholes option pricing formula.**

I. INTRODUCTION

Last year aroused discussion about price of options with deep out of the money characteristics [1]. Some experts believe that selling insurance on some asset price (selling options) and selling lottery tickets have delivered positive long – run rewards in a wide range of investment contexts. Financial markets are full of strategies that resemble insurance or lotteries. Thus, the main question whether investors can boost longterm expected returns by buying or selling insurance and lottery tickets—that is, financial investments that have insurance-like or lottery-like characteristics. The answer depends on the market pricing of skewness: how investors trade asymmetry against mean return. If most investors prefer positive skewness, investments with positively asymmetric payoffs tend to be highly priced and offer relatively low long-run returns. Selling volatility on either the left tail (insurance) or the right tail (lottery tickets) adds value in the long run. Conversely, buying option-based tail risk insurance against financial catastrophes and then holding lottery-like high-volatility investments results in poor long-run returns. As it shown in the Figure 1, implied volatility, which is part of an option price mostly, is overvalued comparing to real volatility. So, it gives possibility to earn positive return on selling options contracts in the long term. However, this outcome is not a foregone conclusion. Conceivably, for a given mean return, investors may prefer to suffer losses in one big hit instead of a slow trickle, implying a preference for negative skewness.

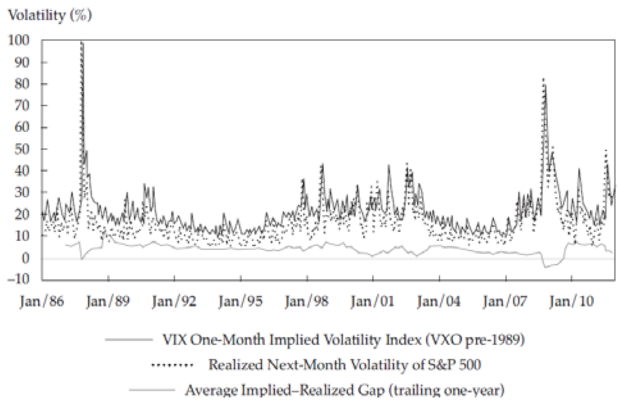

Fig. 1. Historical implied volatility and real volatility

From other hand Table [2] pay attention to huge losses missing nonlinear effects and asymmetries in errors (deviations from the model result in considerably more harm when one is wrong than when one is right). Merely adding these nonlinear responses to tail events does more than reverse the result.

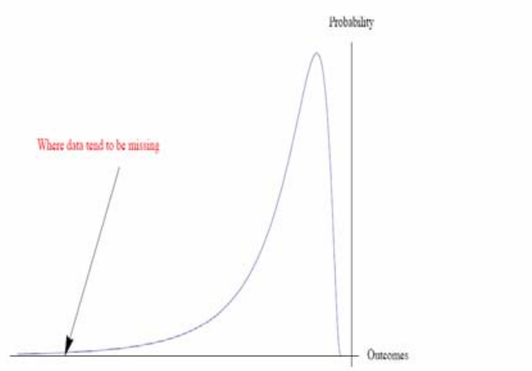

Fig. 2. Distribution of historical returns.

If historical returns are skewed to the left (Figure 2), most missing observations are in the left tails, causing an overestimation of the mean. The more skewed the payoff, and the thicker the left tail, the worst the gap between observed and true mean

Ilmanen [1] included a review of all supporting arguments against the purchase of small- probability events, refuting his article allow the refutation of the prevailing arguments that posit the overpricing of small odds in finance. So, it turns out that there is not a single study that convincingly demonstrates the overpricing of small probabilities in finance or economics.

Hence, one would need more than 2,000 years of data showing an absence of 1987- style crashes - generously assuming that the environment is stable - to pronounce the sale of these options "safe." Even those options that are closer to the money (and commonly traded) deliver large enough a payoff to forbid us from making claims from few decades worth of data; for instance an option 12 standard deviations away from the money return 5,000 the daily erosion. Another misunderstanding concerns the path dependence of these payoffs, which compounds the payoff asymmetry. When the implied volatility quadruples, a 15-standard-deviation out-of-the-money option becomes a 4-standard-deviation option and its value is multiplied by 3,300. Implied volatility (as represented by various volatility indices, such as the Chicago Board Options Exchange Volatility Index, or VIX) quadrupled at least six times over the past quarter century. Another word speaking - options that are extremely nonlinear in response to market movement should be studied with caution before selling.

So, further we want to suggest an approach to analyze the payoff and risks of options based on the nonlinearities (GARCH model) in the tails which is shown on Tesla Motors Inc equity price dynamics. The main idea is to use autocorrelation effect in continuous time approximation of stochastic difference equation in a form of diffusion approximation with stochastic volatility which described by GARCH (1,1) process.

The paper is structured as follows. Section 2 gives a brief review of the proposed model which take into account autocorrelation problem in asset returns and the idea of the diffusion approximation [3]. Section 3 describes Stock Volatility Simulation via Monte Carlo. In Section 4 we report our test results for the distribution of the stationary solution and time to convergence to stationary solution based on fixed parameters, but Section 5 concludes and discusses several possible avenues of the future research.

II. Review of the Proposed Model and Parameters Evaluation

A lot of papers study serial correlation of stock returns. One of the first researches - Mezin [4] found a relationship between asset price volatility, asset return volatility and asset return autocorrelation coefficient. The analytical solution obtained reduces to the well known Black Scholes option pricing formula for the special case of autocorrelation in asset returns. In Mezin's paper were created a framework of a log normally distributed asset price *S* with serially correlated returns and derived an analytic option pricing model, capable of providing an exact solution for a value of derivatives on such an asset. Developed framework of random, normally distributed, process x, such that $lnS = X$, with autocorrelated increments ξ that have volatility σ^2 and autocorrelation coefficient ρ. Both parameters can be estimated using historical data. Instead of heuristically based approach it is possible to assume limit theorems which proposed Carkovs [5] and get the continuous time approximation of stochastic difference equation in a form of the diffusion approximation.

The simplest mathematical model describing development of stock's price S_t and involving assumption of serial autocorrelation in stock's returns under commonly used condition on risk neutrality of probabilistic measure P may be written in the following way

$$S_{t+1} = S_t(1+\varepsilon^2\mu + \varepsilon\sigma y_{t+1}) \tag{1}$$

where y_t is a Gaussian random sequence with zero mean and unit variance. When it is considered that these random numbers are independent we may write that y_t follows AR (1):

$$y_{t+1} = \rho y_t + \sqrt{1-\rho^2}\,\xi_{t+1} \tag{2}$$

where $\{\xi\}_t$, $E\xi_t = 0$, $E\xi_t^2 = 1$ is i.i.d. Gaussian sequence. To be able use results derived by Carkovs in [5] we denote $x_t \equiv S_t$ and rewrite equation (1) in the following form

$$x_{t+1} = x_t + \varepsilon\sigma y_{t+1}x_t + \varepsilon^2\mu x_t \tag{3}$$

These results states that for small ε, equation (3) can be approximated by distribution of the vector $\{X(t_1), X(t_2), \ldots, X(t_n)\}$ defined by solution of stochastic Ito differential equation:

$$dX(s) = a(X(s))ds + \sigma(X(s))d\omega(s) \tag{4}$$

And calculation of (4) we derive continuous time approximation of stochastic difference equation (1) in a form of diffusion process satisfying stochastic Ito differential equation:

$$dS(t) = S(t)(\mu+\sigma^2 k)dt + S(t)\sqrt{1+2k}\,\sigma d\omega(t) \tag{5}$$

$$k := \sum_{m=1}^{\infty} Corr\{y_{t+m}, y_t\} = \frac{\rho}{1-\rho}$$

After substitution we get the final equation

$$dS(t) = S(t)(\mu+\sigma^2\frac{\rho}{1-\rho})dt + S(t)\sqrt{\frac{1+\rho}{1-\rho}}\sigma d\omega(t) \tag{6}$$

This equation is used as a main source for option price derivation using Black Scholes technique. But from other hand we have here constant volatility σ . It is not always true and if we want to get more reliable option price we should give possibility for changing σ in time. For this purpose we can use a very well known GARCH(1,1) process. And further, option price revaluation will be done including mode of the last trajectory value of the Monte Carlo simulation of the volatility process for one year period and derived Black-Scholse option equation with autocorrelation in Tesla Motors Inc returns.

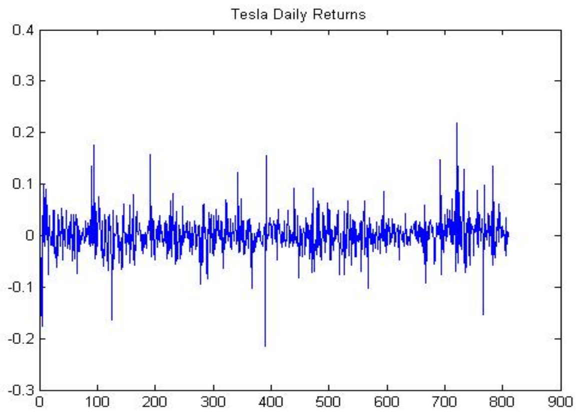

Fig. 3. Tesla Motors Inc daily returns for period from 26.06.2010 till 17.09.2013

III. Stock Volatility Simulation via Monte Carlo

To make option price revaluation taking into consideration above mentioned equation (6) we have to find volatility value for replacing implied volatility which is quoted in the market. Our aim is to find volatility for one year period. Making standard econometrical approach for time series analysis (ACF, PACF, model error tests) were suggested to use for Tesla Motors Inc stock returns volatility simulation GARCH(1,1) model. As we know from [6] that GARCH(1,1) has stochastic diffusion approximation in a form:

$$d\sigma_t^2 = (\omega + (\alpha^2\kappa - \theta)\sigma_t^2)dt + \alpha\sqrt{1+2\kappa}\sigma_t^2 dW(t) \qquad (7)$$

$$C(k) = \mathrm{E}\{y_t y_{t+k}\} \text{ for } k \in \mathbb{N} \qquad \kappa := \sum_{k=1}^{\infty} C(k) \neq 0$$

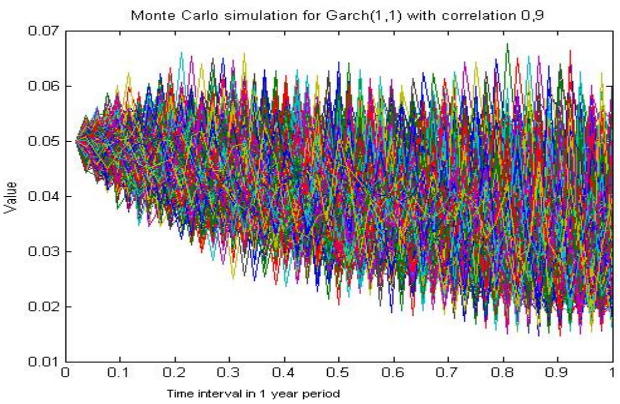

Fig. 4. Volatility paths simulation (1000) results for 1 year period

Parameters α = 0.14011, θ = 0.53541, ω = 0.0004499 were found in MatLab for simple GARCH(1,1) volatility difference equation. So, if parameters are known and assuming that equations (7) correlation parameter k = 9 (correlation is equal to 0.9) , rewriting (7) in a discrete form it is possible to make Monte Carlo simulations (1000 paths, 1year length) in MatLab making simple code for this procedure (see results in figure 4). Volatility process in Figure 4 is decreasing. It gives possibility to take into consideration a lower volatility then market implies for the moment.

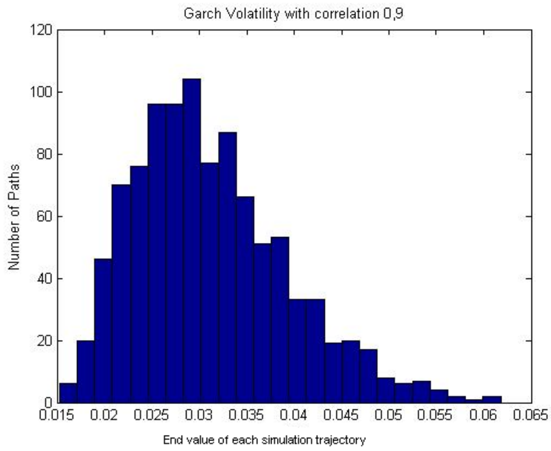

Fig. 5. Distribution of the last value of the volatility Monte Carlo simulation

As it is shown in Figures 5 expected variance distribution is skewed to the right with most possible values in a 25% area. For the Tesla Motors Inc European call option price calculation we can take volatility with a buffer 10%.

IV. Tesla Motors Stock's Option Price Evaluation

Now we can derive European call option pricing formulas if underlying stock's price process *S(t)* satisfies the stochastic differential equation (6). The boundary conditions for the European call option are given in a form $C(S(T),T) = \max(S(t) - K;0)$ and $C(0,t) = 0$ (where K is option strike and T time till option expiration). Using well known techniques we get the following results

$$C(S(t),t) = S(t)N(d_1) - K\exp(-(\mu + \sigma^2 k)(T-t))N(d_2) \qquad (8)$$

where

$$d_1 = \frac{\log(S(t)/K) + (\mu + \sigma^2 k + \frac{1}{2}\sigma^2(1+2k))(T-t)}{\sigma\sqrt{(1+2k)(T-t)}}$$

and

$$d_2 = d_1 - \sigma\sqrt{(1+2k)(T-t)}$$

where N() is the standard normal cumulative distribution function.

Now by replacing results in (8) with evaluated via Monte Carlo volatility with buffer 35% and market volatility as well other conditions for annual interest rate (1% p.a.), strike 380 USD and maturity we get revaluated price for Tesla Motors Inc stock with out of the money Europen call option.

The results in the table can help to make decision for a security portfolio manager, risk analyst or any other person who involved in to option selling. As we see from the table, revaluated option price with small serial correlation in Tesla Motors Inc stock returns is cheaper than it is quoted in the market. But it is not true if we assume large correlation. In a small serial correlation evidence equation (6) with GARCH (1,1) process for volatility suggests to sell European call option on Tesla Motors Inc, because it is overpriced.

Tesla Motors, Inc, Option Valuation		
Option Price	**Black - Scholes**	**Our Approach**
Maturity	1 year	1 year
Strike	380 USD	380 USD
Volatility	57%	35%
Annual, %	1%	1%
Price	5,22 USD	1,36 USD; if ρ =0.2
		102,5 USD; if ρ =0.9

V. Conclusions and Further Work

We have been obtained European call option price for Tesla Motors, Inc, taking into account serial correlation of returns and volatility. This results show that even small levels of predictability due to serial correlation can give substantial deviation from results obtain by Black-Sholes formula. So, selling European call option could be profitable if correctly estimated serial correlation in returns and volatility, as it shown in Tesla Motors Inc case (ACF function is minimal) and Ilmanen [1] hypothesis cannot be rejected.

Further approach in this area could be divided in to 2 different ways. One way is to find a solution of the stochastic differential equation system where volatility is stochastic (see equation 9). It gives the possibility to use Monte Carlo simulations for evaluation possible future stock price outcomes.

$$dy(t) = c\sigma^2 dt + \sigma\sqrt{\frac{1+\rho}{1-\rho}}d\omega_1(t)$$

$$d\sigma^2(t) = \left(\omega - \sigma^2\left(\theta - \alpha^2\frac{\rho^2}{1-\rho^2}\right)\right)dt + \sigma^2\alpha\sqrt{\frac{1+\rho}{1-\rho}}d\omega_2(t) \quad (9)$$

Another uncovered division in option price revaluation is taking copula based semi parametric regressions.

$$t \in Z: X_t = f(X_{t-1},\varepsilon) + g(X_{t-1},\varepsilon)\xi_t \quad (10)$$

This allows to built copulas for GARCH process (we know from our example that Tesla Motors Inc stocks has GARCH effect) and assuming [4] that stationary GARCH(1,1) process has inverse Γ- distribution we can revaluate nonlinear functions parameters. It can be proved that this approach should give the same result because copula density is equal Markov transition probability which helps to find stochastic diffusion approximation for any stochastic difference equation. But make applied calculations with equation (10) is much easier.

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Andrejs Matvejevs, Jegors Fjodorovs. Opciju cenas pārvērtēšana, izmantojot GARCH procesu un tā īpašības
Šajā rakstā tiek aprakstīts opciju cenas pārvērtēšanas mehānisms, izmantojot analītisko risinājumu finanšu aktīvu cenu noteikšanas problēmai ar autokorelēto ienesīgumu. Par darba pamatu kalpo divu autoru hipotēzes par apdrošināšanu un loterijas biļetes pirkšanu kontekstā ar finanšu tirgus gaidām [1], [2] un dažādu zinātnieku pētniecības rezultāti [3], [4] un [5], kas tika veltīti akciju indeksa autokorelācijai, kad šajā indeksā tika iekļautas mazās kapitalizācijas kompānijas. Pētījums balstās uz Tesla Motors Inc kompānijas akcijas ienesīguma dinamiku. Rezultāti parāda, ka eksistē pozitīva korelācija un ienesīguma dispersija mainās laikā. Tādējādi, lai aprēķinātu tādas akciju opciju cenas, ekonometristi attīstījuši GARCH tipa modeļus, kuros nosacītā dispersija varētu mainīties laikā. Bet, kad diskrētais modelis satur nenovērojamo mainīgo (piem., nosacīto dispersiju), varētu būt grūti atrast ticamības funkciju nelineārai stohastisko vienādojumu sistēmai, definētai diskrētos intervālos. Nelsons [3] bija pirmais, kas attīstīja nosacījumus, pie kuriem diferenču tipa GARCH modelis konverģē pēc varbūtības uz Ito procesu, kad diskrēto laika intervālu garums samazinās līdz nullei. Savukārt visa iepriekšminētā ideja tika izstrādāta gadījumā, kad aktīvu ienesīgums nav autokorelēts, bet reālajā situācijā šī korelācija pastāv, kas būtiski varētu mainīt novērtējumu. Tādējādi šajā darbā tiek izmantots nepārtraukts difūzijas modelis (kurā pamatā ir J.Carkova darbi par diferenču vienādojumu aproksimāciju ar stohastisko diferenciālvienādojumu, skat. [5]) gadījumā, kad akciju ienesīgums ir sērijveidā autokorelēts, iegūts Eiropas tipa opcijas cenu noteikšanas vienādojums, kurš ņem vērā autokorelāciju. Bez tam minētais vienādojums iekļauj stohastisko svārstīgumu. Šis svārstīgums tiek modelēts ar Monte Karlo paņēmienu, izmantojot GARCH(1,1) procesa difūzijas aproksimāciju diskrētajā laikā. Šīs aproksimācijas realizācija tika īstenota MatLab vidē. Rezultātā tiek izskaitļota Eiropas tipa opcijas cena Tesla Motors Inc kompānijai ar termiņu 1 gads un parādīts, ka opciju cenu noteikšana ir atkarīga no aktīvu korelācijas un svārstīguma vērtības. Tādējādi Eiropas tipa opciju pārdošana varētu nest peļņu gadījumā, ja ir pareizi novērtēta autokorelācija un svārstīgums, kas ļautu būtiski pārvērtēt rezultātus, kas iegūti ar Black-Scholse formulu, un līdz ar to Ilmanena [1] hipotēzi varētu nenoraidīt.

Андрей Матвеев, Егор Фёдоров. Переоценка цены опциона с использованием процесса GARCH и его свойств.
В настоящей статье мы описываем механизм процесса переоценки опционов, используя аналитическое решение финансовой проблемы оценки активов с автокоррелированными доходностями. В основе работы две гипотезы о страховании и покупке лотерейных билетов в контексте финансовых ожиданий рынка [1] и [2], и труды различного периода о поведении финансовых рынков [3], [4] и [6], которые в основном посвящены автокорреляции индекса акций. Экспериментальные результаты показывают, что существует позитивная автокорреляция, когда в индекс включено большое количество малоликвидных акций. Наше исследование основано на динамике доходности акций компании Tesla Motors. Результаты показывают, что существует положительная автокорреляция доходностей акций этой компании, а так же с течением времени изменяется дисперсия. Таким образом, для расчета цен опционов эконометристы разработали GARCH модели, где условная дисперсия изменчива во времени. Но когда дискретная модель содержит ненаблюдаемые переменные (например, условную дисперсию), это может составить трудности в нахождении функцию правдоподобия нелинейного стохастического уравнения системы определенной на дискретных интервалах. Нельсон [3] был первый, кто предложил условия сходимости по вероятности GARCH модели на Ито процесс, когда интервал дискретного времени стремиться к нулю. Вся вышеупомянутая теория разработана на случай отсутствия автокорреляции, что не всегда согласуется с реальностью. Мы разработали непрерывные модели диффузии для случая серийной корреляции доходностей акций (более подробно с принципами аппроксимации и её сходимости можно ознакомиться в работе Е.Царькова [6]) , получили уравнение для цены европейского опциона, которое учитывает автокорреляцию. Кроме того, в это уравнение включается стохастическая волатильность, которая моделируется методом Монте-Карло с помощью диффузионного приближения GARCH(1,1) процесса. Это моделирование было реализовано в среде MATLAB. В результате были рассчитаны различные варианты цены европейского типа опционов для компании Tesla Motors Инк со сроком погашения 1 год и показано, что цена опциона зависит от корреляции активов и значения волатильности. Таким образом, продажа опционов может быть выгодна, если должным образом оценены автокорреляция и волатильность доходностей, что позволяет существенно переоценить результаты, полученные формулой Блэка- Шоулса и, следовательно, гипотеза Илманена [1] может быть не опровергнута.