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PHASE MERGER PRINCIPLE FOR NONSTATIONARY IMPULSE MARKOV DYNAMICAL SYSTEMS

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Abstract. The paper deals with impulse type non-stationary dynamical systems fast switched by step Markov process having m invariant measures. We have derived more simple Markov dynamical system in a form of ordinary differential equation with right part switched by merger homogeneous Markov process with m states and have proved that for sufficiently fast switching this dynamical system approximates initial system.

Key words and phrases. Averaging principle; impulse dynamical systems; merger procedure.

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1 Introduction

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}^t\}, \mathbf{P})$ be filtered probability space and $\{y_\varepsilon(t), t \in \mathbb{R}\}$ be $\mathfrak{F}^t$-adopted right continuous homogeneous Markov process with countable phase space $\mathbb{Y}$ giving by weak infinitesimal operator $Q^\varepsilon = Q_0 + \varepsilon Q_1$, where

$$Q_0\, v(y) := a(y) \sum_{z\in\mathbb{Y}} [v(z) - v(y)]\, p_0(y,\, z), Q_1\, v(y) := \sum_{z\in\mathbb{Y}} [v(z) - v(y)]\, p_j(y,\, z), \tag{1}$$

ε is small positive parameter, $a(y), p_0(y,\, z), p_1(y,\, z)$ are non-negative uniformly bounded on $y, z \in \mathbb{Y}$ functions, and $\sum_{z\in\mathbb{Y}} p_0(y, z) \equiv 1$. By definition [2] Q_ε is closed operator with located in the left complex half-plane $\{\lambda \in \mathcal{C} : \Re\lambda \leq 0\}$ spectrum $\sigma^\varepsilon := \sigma(Q^\varepsilon)$ for any ε. We will assume that defined by weak infinitesimal operator Q_0 embedded Markov chain is decomposable and there exist m such non-intersecting invariant subspaces $\{\mathbb{Y}_j, j = 1, 2, .., m\}$ that

$$\forall k, j = 1, 2, ..., m, k \neq j, \forall y \in \mathbb{Y}_j, \forall z \in \mathbb{Y}_j : \;\; p_0(y,\, z) = 0 \tag{2}$$

but for all $j = 1, 2, ..., m$ reduction of operator Q_0 at the phase space $\mathbb{B}(\mathbb{Y}_j)$ defines ergodic Markov process with spectrum consisting of simple eigenvalue 0 and remaining subset at the half-plane $\{\lambda \in \mathcal{C} : \Re\lambda < -\rho\}$. This assumption permits to confirm that operator Q_0 has 0 as an isolated simple eigenvalue of multiplicity m with the eigenfunctions defined by equalities

$$q_j\,(y) = \begin{cases} 1, & \text{for } y \in \mathbb{Y}_j \\ 0, & \text{for } y \in \mathbb{Y}_k, \;\; k \neq j. \end{cases} \tag{3}$$

The remaining part of spectrum is situated in the half-plane $\{\lambda \in \mathbf{C} : \Re\lambda < -\rho\}$ for some positive ρ. Adjoint operator Q_0^* also has 0 as an isolated eigenvalue of multiplicity m and m invariant probabilistic measures $\mu_k(y)$ with the same supports $\mathbb{Y}_k,\ k = \overline{1,m}$. Using an averaging procedure by the above mentioned invariant measures we can define numbers

$$k,\ j \in \{1,\ ...,m\}:\ \ \gamma(j,k) := \sum_{y\in\mathbb{Y}_k}\sum_{z\in\mathbb{Y}_j} p_1(y,z)\,\mu_k(y) \tag{4}$$

and draw up a matrix

$$\Gamma = \left\{\frac{\gamma(j,k)}{\gamma(k)}\right\}_{j,k=\overline{1,m}}$$

where $\gamma(k) = \sum_{j=1}^{m}\gamma(j,k)$ for each $k = \overline{1,m}$. By definition [2] we may interpret this matrix as a transition probability matrix for Markov chain with set of states $\bar{\mathbb{Y}} = \{1,2,...,m\}$. Matrix Γ can be embedded into Poisson process [3] with intensities of jumps $\gamma(k), k \in \{1,2,...,m\}$. The defined by this procedure Markov process $\{\hat{y}(t),\ t \geq 0\}$ on the phase space $\bar{\mathbb{Y}} = \{1,2,...,m\}$ has weak infinitesimal operator

$$\hat{Q}v(k) := \sum_{j=1}^{m}\gamma(j,k)[v(j) - v(k)] \tag{5}$$

Using one-one equivalence

$$\bar{\mathbb{Y}} = \{1,2,...,m\} \Longleftrightarrow \{\mathbb{Y}_1,\mathbb{Y}_2,...,\mathbb{Y}_m\} := \hat{\mathbb{Y}}$$

we will refer to the above defined Markov process as *merger Markov process on the merger phase space* $\hat{\mathbb{Y}}$.
Now we define projective operator from space $\mathcal{P} : \mathbb{B}(\mathbb{Y})$ to space $\hat{\mathbb{B}}(\mathbb{Y}) := \mathcal{P}\mathbb{B}(\mathbb{Y})$ with basis (3) by equalities:

$$k \in \bar{\mathbb{Y}}, y \in \mathbb{Y}_k:\ \ (\mathcal{P}v)(y) \equiv \sum_{z\in\mathbb{Y}_k} v(z)\mu_k(z) \tag{6}$$

For any function $v \in \mathbb{B}(\mathbb{Y})$ a difference $v - \mathcal{P}v$ is element of the kernel $\mathcal{K}$ of operator $\mathcal{P}$. Therefore [2] for any $v \in \mathbb{B}(\mathbb{Y})$ there exists improper integral:

$$\begin{aligned}(\Pi v)(y):\ &= -\int_0^\infty e^{tQ_0}[v(z) - (\mathcal{P}v)(z)]dt \\ &= \int_t^\infty \mathbf{E}_{t,y}\{v(y(s)) - (\mathcal{P}v)(y(s))\}ds\end{aligned} \tag{7}$$

where $P_0(t,y,z)$ is corresponding to weak infinitesimal operator Q_0 transition probability. Formula (7) defines linear continuous operator $\Pi : \mathbb{B}(\mathbb{Y}) \to \hat{\mathbb{B}}(\mathbb{Y})$. It is not difficult to prove [2] that

$$k \in \bar{\mathbb{Y}}, y \in \mathbb{Y}_k:\ \ Q_0(\Pi v)(y) = (\mathcal{P}v)(y) - v(y) \tag{8}$$

for any $v \in \mathbb{B}(\mathbb{Y})$.

By definition any realization of Markov process $\{y_\varepsilon(t)\}$ is piecewise constant right continuous function with jumps at time moments $\{\tau_j^\varepsilon, j \in \mathbb{N}\}$. Using this sequence we define *nonstationary fast oscillating impulse Markov dynamical systems in* $\mathbf{R}^d$ as the system of equations:

- *the differential equation in* $\mathbb{R}^d$*:*

$$\frac{dx^{\varepsilon}(t)}{dt} = f(t\varepsilon^{-1}, x^{\varepsilon}(t), y_{\varepsilon}(t\varepsilon^{-1})) \tag{9}$$

  for all $t \in \bigcup_{j\in\mathbb{N}}(\tau_j^{\varepsilon}, \tau_{j+1}^{\varepsilon})$*,*

- *the difference equation in* $\mathbb{R}^d$*:*

$$x^{\varepsilon}(\tau_j^{\varepsilon}) = x^{\varepsilon}(\tau_{j-1}^{\varepsilon}) + \varepsilon g(\varepsilon^{-1}\tau_{j-1}^{\varepsilon}, x^{\varepsilon}(\tau_{j-1}^{\varepsilon}), y_{\varepsilon}(\varepsilon^{-1}\tau_{j-1}^{\varepsilon})) \tag{10}$$

  for all $j \in \mathbb{N}$*,*

- *initial conditions:*

$$\tau_0 = t_0, y_{\varepsilon}(\varepsilon^{-1}t_0) = y_0, x^{\varepsilon}(t_0) = x_0 \tag{11}$$

We assume that:

(i) functions $f(t, y, x)$ and $g(t, y, x)$ have continuous derivative on$t$ and two continuous derivatives on x;

(ii) there exists such a constant L that

$$\forall x_1, x_2 \in \mathbb{R}^d : \sup_{t,y} [|f(t, x_1, y) - f(t, x_2, y)| + |g(t, x_1, y) - g(t, x_2, y)|] \le L|x_1 - x_2|$$

(iii) there exists such a constant C that

$$\sup_{t,y}\{|f(t, y, x)| + |g(t, y, x)|\} \le C(1 + |x|)$$

(iv) for any $x \in \mathbb{R}^d$, $s \in \mathbb{R}$, and $y \in \mathbb{Y}$ there exist

$$\bar{f}(y, x) := \lim_{T\to\infty} \frac{1}{T} \int_s^{s+T} f(t, y, x)dt, \;\; \bar{g}(y, x) := \lim_{T\to\infty} \frac{1}{T} \int_s^{s+T} g(t, y, x)dt \tag{12}$$

(v) there exists such a constant K that for any $T > 0, y \in \mathbb{Y}$, and $s \in \mathbb{R}$

$$\left|\int_s^{s+T} [f(t, y, x) - \bar{f}(y, x)]dt\right| \le K(1 + |x|), \;\; \left|\int_s^{s+T} [g(t, y, x) - \bar{g}(y, x)]dt\right| \le K(1 + |x|) \tag{13}$$

It is not difficult to prove that for any t_0, y, and x in (11) on the assumptions (i)–(iii) there exists unique stochastic process satisfying (9)-(10)-(11).
The same like in the papers [4] and [5] we may study the satisfying (9)-(10)-(11) triple $\{\varepsilon^{-1}t, y_{\varepsilon}(\varepsilon^{-1}t), x^{\varepsilon}(t)\}$ as Markov process on enlarge phase space $\{\mathbb{R}, \mathbb{Y}, \mathbb{R}^d\}$ with weak infinitesimal operator $\mathcal{L}^{\varepsilon}$ given for sufficient smooth on t and x function $v(t, y, x)$ by equality

$$\mathcal{L}^{\varepsilon}v(t, y, x) = \frac{1}{\varepsilon}\left(\frac{\partial}{\partial t} + Q_0\right) v(t, y, x) + Q_1 v(t, y, x) + \\ + \left(f(t\varepsilon^{-1}, y, x), \nabla_x\right) v(t, y, x) + G^{\varepsilon}v(t, y, x) \tag{14}$$

where $(.,.)$ is scalar product, ∇_x is operator-gradient in $\mathbb{R}^d$, and

$$G^\varepsilon v(t,y,x) = \frac{1}{\varepsilon}\sum_{z\in\mathbb{Y}}[v(t,y,x+\varepsilon g(\varepsilon^{-1}t,y,x)) - v(t,y,x)](a(y)p_0(y,z)+\varepsilon p_1(y,z)) \tag{15}$$

Let $\mathbb{V}$ be set of satisfying assumptions (12) - (13) bouded continuous differentiable by t and x functions $v(t,y,x)$. For any $v \in \mathbb{V}$ after averaging by t

$$\bar{v}(y,x) := \lim_{T\to\infty}\frac{1}{T}\int_0^T v(t,y,x)dt$$

we may define operator

$$\begin{aligned}(\mathcal{R}v)(t,y,x) : &= \int_0^t [\mathcal{P}v(s,y,x) - \mathcal{P}\bar{v}(y,x)]ds + \\ &+ \int_t^\infty e^{-Q_0(t-s)}[v(s,y,x) - \mathcal{P}v(s,y,x)]ds \end{aligned}\tag{16}$$

Applying formula (8) it is not difficult to prove that for any $v \in \mathbb{V}$:

$$\left(\frac{\partial}{\partial t} + Q_0\right)\mathcal{R}v = \mathcal{P}\bar{v} - v \tag{17}$$

After averaging by t and execution by operator $\mathcal{P}$ from (6) the vector-function

$$\psi(t,y,x) := f(t,y,x) + a(y)g(t,y,x)$$

we can define *merger process* $\bar{x}(t)$ as solution of equation

$$\frac{d\bar{x}(t)}{dt} = F(y(t),\bar{x}(t)) \tag{18}$$

where $F(y,x) := \mathcal{P}\{\bar{\psi}(y,x)\}$. As $F(y,x)$ remains unchanged by y at any $\hat{\mathbf{Y}}_k$ we may look for merger process as solution of more simple *merger equation*:

$$\frac{d\hat{x}(t)}{dt} = \hat{F}(\hat{y}(t),\hat{x}(t)) \tag{19}$$

where $\{\hat{y}(t)\}$ is above defined Markov process with infinitesimal operator $\hat{Q}$ on the phase space $\bar{\mathbb{Y}} = \{1,2,...,m\}$. Defined by equations (18) and (19) processes have the same distributions because for any $k \in \{1,2,...,m\}, y \in \mathbb{Y}_k$, and $x \in \mathbb{R}^d$ we have identity $F(y,x) \equiv \hat{F}(k,x)$.

Theorem 1.1 [Merger principle] *Under assumptions (i)–(v) the defined by* (9)-(10)-(11) *family of processes* $\{\{x^\varepsilon(s)), s \in [t_0, t_0+T]\}, \varepsilon > 0\}$ *for any* $t_0 \in \mathbb{R}$ *and* $T > 0$ *weak converges with* $\varepsilon \to 0$ *to the solution of equation* (18) *with corresponding initial conditions.*

Proof. Later we will operate with functions $\varphi(t,y,x,u) = \psi(t,y,x) - F(y,u)$ and $\mathcal{R}\varphi(t,y,x)$. Owing to the assertions (i)–(v) this functions the same as functions $\psi(t,y,x)$ and $F(y,x)$ are bounded continuous differentiable by x and have sub-linear growth by x at infinity, that is, there exists such a constant $a > 0$ that

$$|\psi(t,y,x)| \le a(1+|x|), |F(y,u)| \le a(1+|u|) \ \ |\mathcal{R}\varphi(t,y,x,u)| \le a(1+|x|+|u|) \tag{20}$$

for all $t \in \mathbb{R}, y \in \mathbb{Y}$. To analyze an asymptotic by ε of satisfying (9)-(10)-(11) stochastic process we define

- function $v_1(t,y,x,u) = 2(x-u,\ (\mathcal{R}\varphi)(\varepsilon^{-1}t,\ y,\ x,\ u))$, which owing to formula (20) satisfies inequality

  $$|v_1(t,y,x,u)| \le a(1+|x|^2+|u|^2) \tag{21}$$

  with some positive constant a;

- function

  $$v^\varepsilon(t,x,\ y,\ u) := |x-u|^2 + \varepsilon v_1(t,y,x,u) + \varepsilon(a+1)(1+|x|^2+|u|^2) \tag{22}$$

  where constant a is taken from (21);

- compound Markov process $\{t, x^\varepsilon(t), y_\varepsilon(t), \bar{x}(t)\}$ with weak infinitesimal operator $\mathbf{L}(\varepsilon)$, which executes on function (22) in the following way:

  $$\mathbf{L}(\varepsilon)v^\varepsilon(t,y,x,u) = \mathcal{L}^\varepsilon v^\varepsilon(t,y,x,u) + (F(u,y), \nabla_u)v^\varepsilon(t,y,x,u) \tag{23}$$

Applying formulae (20) and (21) we can compute that

$$|x-u|^2 + \varepsilon(1+|x|^2+|u|^2) \le v^\varepsilon(t,x,\ y,\ u) \le |x-u|^2 + \varepsilon(2a+1)(1+|x|^2+|u|^2) \tag{24}$$

Not so difficult to prove that function $v_1(t,y,x,u)$ has bounded continuous derivative on x and u. Therefore we may write an equality

$$\begin{aligned}\frac{1}{\varepsilon}[v_1(t,y,x+\varepsilon g(\varepsilon^{-1}t,y,x),u) - v(t,y,x,u)] &= \big(g(t\varepsilon^{-1},y,x),(\nabla_x v_1)(t,y,x,u)\big) + \\ + \big(g(t\varepsilon^{-1},y,x),(\nabla_x v_1)(t,y,x+\varepsilon\theta(t,y,x),u)\big) &- \big(g(t\varepsilon^{-1},y,x),(\nabla_x v_1)(t,y,x,u)\big)\end{aligned} \tag{25}$$

where $\theta_\varepsilon(t,x,y,u) \in [0,1]$. This assertion permits to rewrite formula (23) in a form of equality:

$$\begin{aligned}\mathbf{L}^\varepsilon v(t,y,x) &= \frac{1}{\varepsilon}\left(\frac{\partial}{\partial t} + Q_0\right)v(t,y,x) + Q_1 v(t,y,x) + (F(u,y),\nabla_u)v_1(t,y,x,u) + \\ &+ \ \big(\psi(\varepsilon^{-1}t,y,x),\nabla_x\big)\,v(t,y,x) + \varepsilon h(t,y,x,\varepsilon)\end{aligned}$$

where function $h(t,y,x,\varepsilon)$ satisfies inequality $|h(t,y,x,\varepsilon)| \le C(1+|x|^2+|u|^2)$ with some positive constant C uniformly on t,y,x,u, and $\varepsilon \in [0,1]$. Owing assertions (9)-(10)-(11) and independence of function $|x-u|^2$ on t and y we can rewrite the above formula in a form of equality:

$$\begin{aligned}\mathbf{L}(\varepsilon)v^\varepsilon(t,y,x,u) &= 2\big(x-u,\psi(\varepsilon^{-1}t,y,x)) - F(u,y)\big) + \\ &+ \ 2\left(x-u,\ \left(\frac{\partial}{\partial t} + Q_0\right)(\mathcal{R}\varphi)(\varepsilon^{-1}t,\ y,\ x,\ u)\right) + \\ &+ \ \varepsilon h(t,y,x,u,\varepsilon)\end{aligned} \tag{26}$$

Therefore on the ground of formulae (17) and (24):

$$\mathbf{L}(\varepsilon)v^\varepsilon(t,y,x,u) = \varepsilon h(t,y,x,u,\varepsilon) \le \varepsilon C(1+|x|^2+|u|^2) \le Cv^\varepsilon(t,y,x,u) \tag{27}$$

Let process $\{x^\varepsilon(t), t\in[t_0,t_0+T]\}$ be solution of (9)–(10)–(11) and process $\{\bar{x}(t), t\in[t_0,t_0+T]\}$ be solution of equation (18) satisfying initial condition $\bar{x}(t_0)=x$. By definition [1] of weak infinitesimal operator $\mathbf{L}(\varepsilon)$ we can derive inequality:

$$\begin{aligned}&\frac{d}{dt}\mathbf{E}\{e^{-Ct}v^\varepsilon(t,y(t),x^\varepsilon(t),\bar{x}(t))\} = -Ce^{-Ct}\mathbf{E}\left\{v^\varepsilon(t,y_\varepsilon(\varepsilon^{-1}t),x^\varepsilon(t),\bar{x}(t))\right\} + \\ + \quad & e^{-Ct}\mathbf{E}\left\{\left(\mathbf{L}(\varepsilon)v^\varepsilon\right)(t,y_\varepsilon(\varepsilon^{-1}t),x^\varepsilon(t),\bar{x}(t))\right\} \le 0\end{aligned} \tag{28}$$

for all $t\in[t_0,t_0+T)$. Therefore

$$e^{-Ct}\mathbf{E}\{v^\varepsilon(t,y_\varepsilon(\varepsilon^{-1}t),x^\varepsilon(t),\bar{x}(t))\} \le e^{-Ct_0}v^\varepsilon(t_0,y_0,x_0,x_0) \tag{29}$$

for all $t\in[t_0,t_0+T]$, and in addition

$$v^\varepsilon(t_0,y_0,x_0,x_0) = \varepsilon h(t_0,y_0,x_0,x_0) \le \varepsilon C(1+2|x_0|^2)$$

because processes $x^\varepsilon(t)$ and $\bar{x}(t)$ have the same initial condition x_0. Applying inequality (24) we can derive for any $\lambda>0$ inequality

$$\begin{aligned}\mathbf{P}\{|x^\varepsilon(t)-\bar{x}(t)|\ge\lambda\} &\le \frac{1}{\lambda^2}\mathbf{E}\{|x^\varepsilon(t)-\bar{x}(t)|^2\} \le \frac{1}{\lambda^2}\mathbf{E}\{v^\varepsilon(t,y_\varepsilon(\varepsilon^{-1}t),x^\varepsilon(t),\bar{x}(t))\} \le \\ &\le \varepsilon\frac{C}{\lambda^2}e^{C(t-t_0)}(1+2|x_0|^2) \le \varepsilon\frac{C}{\lambda^2}e^{CT}(1+2|x_0|^2)\end{aligned}$$

for all $t\in[T-0,t_0+T]$. After limit processing

$$\lim_{\varepsilon\to0}\sup_{t_0\le t\le t_0+T}\mathbf{P}\{|x^\varepsilon(t)-\bar{x}(t)|\ge\lambda\} \le \lim_{\varepsilon\to0}\left\{\varepsilon\frac{C}{\lambda^2}e^{CT}(1+2|x_0|^2)\right\} = 0$$

proof is completed.

References

[1] J.L.Doob, *Stochastic processes*, John Willey & Sons, New York, 1953.

[2] E. B. Dynkin, *Markov Processes*, Academic Press, New York, 1965.

[3] V.S. Koroliuk, A. F. Turbin, *Mathematical Foundations of the State Lumping of Large Systems*, Springer Verlag, New York, 1993.

[4] A.Pola, . *Averaging of impulse systems with Markov switchings and fast time*, Proceedings of the Latvian Probability Seminar, **4** (1996), 9–18.

[5] Ye. Tsarkov (J.Carkovs), *Averaging in Dynamical Systems with Markov Jumps*, 1993, Preprint No. 282, April, Institute of Dynamical Systems, Bremen University, Germany.

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