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MARKOV SWITCHED DIFFERENCE EQUATIONS OF INVESTMENT RISK ANALYSIS

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Abstract. This paper deals with linear difference equations with coefficients dependent on the Markov chain. We derive operator equations for the first and the second moments, which can be used by investors for Markowitz approach to portfolio selection. This results the most convenient for equations with near to constant coefficients. In this case we have succeeded in asymptotic approximation of the above mentioned operator equations by an ordinary difference equations with constant coefficients. Besides we have proved that covariance matrices of solutions can be analyzed as powers of a positive operator in partially ordered Banach space. This permits to formulate the necessary and sufficient mean square stability condition as a spectral problem. The proposed method gives very convenient criterion for ergodic property analysis of GARCH(p,q) processes. Much of our paper is devoted to asymptotic analysis of difference equations subjected to small parameter. For these equation we derive diffusion approximation method and prove that resulting stochastic Ito equation may be used for stationary solution analysis. In order to illustrate of this method availability we analyze the most popular GARCH(1,1) model for portfolio volatility. Our proposal approach permits to develop diffusion approximation for regression model with correlated residuals and to discuss dependence of stationary volatility on correlation coefficient.

Key words: Stochastic difference equations, Markov dynamical systems, mean square stability.

Mathematics Subject Classification: Primary 60H10, 60H30; Secondary 37H10.

1 Introduction

The main problem of financial engineering is creation of new compound financial instruments using as a basis such initial securities as bonds, stocks, and other assets. Generated at present time t derivative securities are sharing in speculations at any time moments $t + h > t$ and the later price of these derivatives are dependent on prices of involved initial securities. Most derivatives we may display as a portfolio

$$P(t) = \sum_{k=1}^{n} a_k S_k(t) \tag{1}$$

of initial securities $S_k(t), k = 1, 2, ..., n$. To manage an investment portfolio at initial time t for the purpose of later profit a manager needs an information on dynamics of securities prices

$\{S_k(u)\}$ for all $u > t$ and $k = 1, 2, ..., n$. Usually security dynamics may be modeled as piecewise constant stochastic process with uniform time intervals $t_{j+1} - t_j = h$ of constancy and jumps defined by formula

$$S_k(t_{j+1}) = \sum_{m=1}^{n} r_{km}(t_{j+1}) S_m(t_j) \tag{2}$$

where $r_{km}(t_{j+1})$ are random interest rates dependent on market uncertainty. According to mean-variance Harry Markowitz theory [15] an investor either would like to maximize his portfolio return for a given level of risk, which is proportional to portfolio variance. At any time moment t his strategy is based on expected returns (mean) and the standard deviation (or variance) of the portfolio. Therefore to take a decision on portfolio structure $\{a_1, a_2, ..., a_n\}$ a stag needs equations for first moment $\mathbf{E}\{\vec{S}(t)\}$ and covariance matrix $\mathbf{E}\{\vec{S}(t)(\vec{S}(t))^T\}$ of security vector $\vec{S}(t) = colon\{S_j(t), j = 1, 2, ..., n\}$. This problem will be discussed in the second and the third section of our lecture. There we will deal with the system of linear difference equations in $\mathbb{R}^n$:

$$x_t = A(y_t)x_{t-1}, \ \ t \in \mathbb{N} \tag{3}$$

where $\{A(y), \ y \in \mathbb{Y}\}$ is uniformly bounded continuous $n \times n$ matrix function and $\{y_t, \ t \in \mathbb{N}\}$ is homogeneous Markov chain with transition probability $p(y, dz)$ on the metric space $\mathbb{Y}$. We will derive equation for the conditional first moment $m_t(y) := \mathbf{E}\{x_t/y(t) = y\}$ in the space $\mathbb{C}_n(\mathbb{Y})$ of continuous vector-functions. This permits to analyze unconditional first moments $M_t = \mathbf{E}\{x_t\}$ applying averaging by invariant measure $\mu(dy)$ of ergodic Markov process: $M_t = \int_{\mathbb{Y}} m_t(y)\mu(dy)$. The same results for the second moments $Q_t = \mathbf{E}\{x_t(x_t)^T\}$ for solution of equation (3) will be examined in the third section. More convenient for application result will be derived for equations (3) with near constant coefficients: $A(y) := A_0 + \sum_{k=1}^{m} \varepsilon^k A_k(y)$, where ε is small positive parameter. The derived there results are very important for analysis of the most popular mathematical model [17] for the log of cumulative excess returns $Y_k := \ln S(t_k)$ given as AR(1) regressive equation $Y_{k+1} = a + bY_k + \varepsilon_k$ with stationary residuals having non-constant conditional variances $\{\sigma_k^2 := \mathbf{E}\{\varepsilon_{k+1}^2/\mathfrak{F}^k\}$. According to proposed by [9] approach these variances satisfies scalar difference equation (GARCH(p,q) equation).The derived in the third section methods and results permit to analyze stability problem for stationary volatility using specially constructed inequalities involving GARCH(p,q) coefficients.

The forth section deals with d-dimensional difference equations in $\mathbb{R}^d$

$$x_{t+1} = x_t + \varepsilon f_1(x_t, y_t) + \varepsilon^2 f_2(x_t, y_t), \tag{4}$$

where right part depends on small positive parameter ε. Taking the length of time intervals between observations $t_{k+1} - t_k := h$ more and more finely we derive stochastic approximation for the above difference equations as the system of stochastic differential Ito equations. These results we will apply to asymptotic analysis of the most popular model for the log of cumulative excess returns Y_t on a portfolio [17] with volatility as GARCH(1,1) process.

2 First moments of linear Markov switched difference equations

Let us assume that involved in (3) Markov chain $\{y_t \in \mathbb{Y}, t \in \mathbb{Z}\}$ with transition probability $p(y, dy)$ is given on filtered probabilistic space $(\Omega, \mathfrak{F}, \{\mathfrak{F}^t\}, \mathbf{P})$ and possess Feller property [7]:

$$\forall v \in \mathbb{C}(\mathbb{Y}): \quad (\mathcal{P}v)(y) = \int\limits_{\mathbb{Y}} v(z)p(y, dz) \in \mathbb{C}(\mathbb{Y}) \tag{5}$$

For simplicity, we will suppose that $\mathbb{Y} \subset \mathbb{R}$ and matrix-function $A(y)$ is bounded. The main assumption, which we need in this paper is exponential ergodic property of above defined Markov chain:

- *there exists such number $\gamma \in (0,1)$, that spectrum $\sigma(\mathcal{P})$ of operator $\mathcal{P} : \mathbb{C}(\mathbb{Y}) \to \mathbb{C}(\mathbb{Y})$ may be presented in a form:*

$$\sigma(\mathcal{P}) = \{1\} \bigcup \sigma_\gamma, \; \sigma_\rho \subset \{\lambda \in \mathcal{C} : |\lambda| < \gamma\} \tag{6}$$

Owing this assumption there exists [7] unique invariant probabilistic measure $\mu(dy)$. Using this measure we may define Gilbert space $\mathbb{G} := \mathbb{L}_2^d(\mu)$ of square-integrable vector-functions with scalar product

$$< \mathbf{u}, \mathbf{v} >:= \int\limits_{\mathbf{Y}} (u(y), v(y))\mu(dy)$$

where is scalar product in $\mathbf{R}^d$, that is, $(u, v) := v^u$. The solution of equation (3) with initial initial conditions $x_k = x, y_k = y$ is defined as the sequence

$$\forall x_m(k, x, y) = A(y_m)A(y_{m-1}) \cdots A(y_{k+1})x := \mathbf{X}_k^m x, m > k \tag{7}$$

By definition we will set $\mathbf{X}_k^k = I$ for any $k \in \mathbb{Z}$. To derive equation for the first moment we introduce on the space $\mathbb{G}$ linear continuous operator $\mathbf{A} : \mathbb{G} \to \mathbb{G}$:

$$\mathbf{u} \in \mathbb{C}_d(\mathbb{Y}), y \in \mathbb{Y} : (\mathbf{A}\mathbf{u})(y) := \int\limits_{\mathbb{Y}} A^T(z)\mathbf{u}(z)p(y, dz) \tag{8}$$

Lemma 2.1 *For all $s \in \mathbb{Z}$,$t > 0$, $\mathbf{u} \in \mathbb{G}$, and $\mathbf{x} \in \mathbb{G}$*

$$\mathbf{E}\{(X_s^{s+t}\mathbf{x}(y_s), \mathbf{u}(y_{s+t}))/\mathfrak{F}^s\} = (\mathbf{x}(y_s), (\mathbf{A}^{\mathrm{t}}\mathbf{u})(y_s)) \tag{9}$$

where $(x, v) := v^T x$ is scalar product in $\mathbb{R}^d$.

Proof. For $t = 1$ by Markov property we may write

$$\mathbf{E}\{(X_s^{s+1}\mathbf{x}(y_s), \mathbf{u}(y_{s+1}))/\mathfrak{F}^s\} = \mathbf{E}\{(\mathbf{x}(y_s), A^T(y_{s+1})\mathbf{u}(y_{s+1}))/y_s\} = (\mathbf{x}(y_s), (\mathbf{A}\mathbf{u})(y_s))$$

And now to prove formula (9) we will use induction condition: if (9) is true for $t = m$ then

$$\begin{aligned}
&\mathbf{E}\{(X_s^{s+m+1}\mathbf{x}(y_s), \mathbf{u}(y_{s+m+1}))/\mathfrak{F}^s\} = \\
&\mathbf{E}\{\mathbf{E}\{(X_s^{s+m}\mathbf{x}(y_s), A^T(y_{s+m+1})\mathbf{u}(y_{s+m+1}))/\mathfrak{F}^{s+m}\}/\mathfrak{F}^s\} = \\
&\mathbf{E}\{\mathbf{E}\{(z, A^T(y_{s+m+1}))\mathbf{u}(y_{s+m+1}))/\mathfrak{F}^{s+m}\}\big|_{z=X_s^{s+m}\mathbf{x}(y_s)}/\mathfrak{F}^s\} = \\
&\mathbf{E}\{(X_s^{s+m}\mathbf{x}(y_s), (\mathbf{A}\mathbf{u})(y_{s+m}))/\mathfrak{F}^s\} = (\mathbf{x}(y_s), (\mathbf{A}^{\mathrm{m}}(\mathbf{A}\mathbf{u}))(y_s)) = (\mathbf{x}(y_s), (\mathbf{A}^{\mathrm{m+1}}\mathbf{u})(y_s))
\end{aligned}$$

Proof is complete.
This lemma permits to find all coordinates of the first moment in a form $\mathbf{E}\{x_{s+t}(s,x,y)\} := \mathbf{m}_t(y)$ applying (9) to constant vectors $\mathbf{e}_j, j = 1,2,...,d$ of the unit bases matrix $I = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d\}$ in $\mathbb{R}^d$:

$$\mathbf{m}_t(y) := \mathbf{E}\{x(s+t,s,x,y)\} := \mathbf{E}\{(X_s^{s+t}x/y_s = y\} = (\mathbf{A}^{\mathrm{t}}I)^T(y)x \tag{10}$$

In reality vector-function $\mathbf{m}_t(y)$ we can find only for discrete phase space $\mathbb{Y} = \{a_1, a_2, ..., a_n\}$ with sufficiently small n. This section proposes a working approximation $\bar{\mathbf{m}}_t$ of (10) for equation

$$x_t = A(y_t, \varepsilon)x_{t-1}, \ \ t \in \mathbb{N} \tag{11}$$

with near to constant matrix $A(y,\varepsilon) = A_0 + \sum_{k=1}^{m} \varepsilon^k A_k(y)$. Operator $\mathbf{A}(\varepsilon)$ for this equation may be presented in a following form:

$$\mathbf{A}(\varepsilon) = \mathbf{A}_0 + \sum_{k=1}^{m} \varepsilon^k \mathbf{A}_k \tag{12}$$

with operators $\mathbf{A}_k$ defined by formulae:

$$\mathbf{u} \in \mathbb{G}, y \in \mathbb{Y}: \qquad (\mathbf{A}_0\mathbf{u})(y) := A_0^T \int_{\mathbb{Y}} \mathbf{u}(z)p(y,dz) \tag{13}$$

$$(\mathbf{A}_k\mathbf{u})(y) := \int_{\mathbb{Y}} A_k^T(z)\mathbf{u}(z)p(y,dz), \ \ k = 1,2,...,m \tag{14}$$

The space of d-dimensional vector function $\mathbb{G}$ we may define [12] as the tensor product $\mathbb{G} := \mathbb{R}^d \otimes \mathbb{L}_2(\mu))$. Therefore $\sigma(\mathbf{A}_0) = \sigma(A_0)$, and assuming

$$\{\lambda_1\lambda_2 : \lambda_1 \in \sigma(A_0), \lambda_2 \in \sigma_\gamma\} \cap \sigma(A_0) = \emptyset \tag{15}$$

we may be sure [12] that for sufficiently small $\varepsilon > 0$ there exists decomposition $\sigma(\varepsilon) = \sigma_0(\varepsilon) \cup \sigma_\gamma(\varepsilon)$ for spectrum set $\sigma(\varepsilon) := \sigma(\mathbf{A}(\varepsilon))$ of operators $\mathbf{A}(\varepsilon)$ with d-dimensional spectrum subset $\sigma_0(\varepsilon)$. Besides

$$\sigma_0(\varepsilon) \cap \sigma_\gamma(\varepsilon) = \emptyset, \lim_{\varepsilon\to 0} \sigma_0(\varepsilon) = \sigma(A_0), \lim_{\varepsilon\to 0} \sigma_\gamma(\varepsilon) = \sigma_\gamma \tag{16}$$

Corresponding to spectral sets $\sigma_0(\varepsilon)$ and $\sigma_\gamma(\varepsilon)$ spectral projectors $\mathbf{P}_0(\varepsilon), \mathbf{P}_\gamma(\varepsilon)$ are commutative with $\mathbf{A}(\varepsilon)$ and $\mathbf{P}_0(\varepsilon)\mathbf{P}_\gamma(\varepsilon) = \mathbf{P}_\gamma(\varepsilon)\mathbf{P}_0(\varepsilon) = 0$. These projectors specify a decomposition of the space $\mathbb{G}$ as a direct sum:

$$\mathbb{G} = \mathbf{P}_0(\varepsilon)\mathbb{G} + \mathbf{P}_\gamma(\varepsilon)\mathbb{G} \tag{17}$$

Therefore we make use of formula

$$\forall \mathbf{u} \in \mathbb{G}: \ \ \mathbf{A}(\varepsilon)\mathbf{u} = \mathbf{P}_0(\varepsilon)\mathbf{A}(\varepsilon)\mathbf{P}_0(\varepsilon)\mathbf{u} + \mathbf{P}_\gamma(\varepsilon)\mathbf{A}(\varepsilon)\mathbf{P}_\gamma(\varepsilon)\mathbf{u} \tag{18}$$

Besides the above defined projectors are analytical operator-functions on ε and by definition for any vector constant $x \in \mathbb{R}^d$

$$\lim_{\varepsilon\to 0}(\mathbf{P}_0(\varepsilon)\mathbf{A}(\varepsilon)\mathbf{P}_0(\varepsilon)x)(y) \ = \ (\mathbf{A}_0x)(y) = A_0^T x, \tag{19}$$

$$\lim_{\varepsilon\to 0}(\mathbf{P}_\gamma(\varepsilon)\mathbf{A}(\varepsilon)\mathbf{P}_\gamma(\varepsilon)x)(y) \ = \ 0 \tag{20}$$

Therefore for any $x \in \mathbb{R}^d$ applying equality (10) we may write equation

$$\begin{aligned} \mathbf{m}_t(y) &:= \mathbf{E}\{x(s+t,s,x,y)\} := \mathbf{E}\{(X_s^{s+t}x/y_s = y\} = [(\mathbf{A^t}(\varepsilon)I)(y)]^T x = \\ &= \bar{\mathbf{m}}_t(y) + O(||A_0||^t \gamma^t \varepsilon^t) \end{aligned} \tag{21}$$

where $\bar{\mathbf{m}}_t(y) := [((\mathbf{A}(\varepsilon))^t \mathbf{P}_0(\varepsilon)I)(y)]^T x$. This means that as working approximation $\bar{\mathbf{m}}_t$ of $\mathbf{m}_t$ we may use contraction of operator $\mathbf{A}(\varepsilon)$ on the space $\mathbb{G}_0(\varepsilon) = \mathbf{P}_0(\varepsilon)\mathbb{G}$. We will do that using d noncollinear vector functions

$$(\mathbf{P}_0(\varepsilon)I)(y) := \mathbf{B}(y,\varepsilon) := \{\mathbf{b}_1(y,\varepsilon), \mathbf{b}_2(y,\varepsilon), ..., \mathbf{b}_d(y,\varepsilon)\}$$

as basis in $\mathbb{G}_0(\varepsilon)$. By definition any vector function $\mathbf{u} \in \mathbb{G}_0(\varepsilon)$ is linear combination of basis function. Therefore there exists such $d \times d$-matrix $\Lambda(\varepsilon)$ that

$$\begin{aligned} (\mathbf{A}(\varepsilon)\mathbf{B})(y,\varepsilon) &:= \{(\mathbf{A}(\varepsilon)\mathbf{b}_j)(y,\varepsilon), j = 1,2,...,d\} = \\ &= \left\{\sum_{k=1}^{d} \lambda_{jk}(\varepsilon)\mathbf{b}_k(y,\varepsilon), j = 1,2,...,d\right\} := \mathbf{B}(y,\varepsilon)\Lambda(\varepsilon) \end{aligned} \tag{22}$$

and

$$\mathbf{E}\{x(s+t,s,x,y)\} \approx \bar{\mathbf{m}}_t(y) = (\Lambda(\varepsilon)^T)^t (\mathbf{B}(y,\varepsilon))^T x \tag{23}$$

Owing to decomposition (12) of operator $\mathbf{A}(\varepsilon)$ we may present matrix $\Lambda(\varepsilon)$, and basis matrix $\mathbf{B}(y,\varepsilon)$ in a form of small-parameter ε expansion

$$\Lambda(\varepsilon) = \Lambda_0 + \varepsilon \sum_{k=0}^{\infty} \Lambda_{k+1}, \mathbf{B}(y,\varepsilon) = I + \varepsilon \sum_{k=0}^{\infty} \mathbf{B}_{k+1}(y) \tag{24}$$

Now we can substitute this decomposition in (22) and equate coefficients near appropriate power of ε:

$$A_0^T = \Lambda_0 \tag{25}$$

$$A_0^T \mathcal{P}\{\mathbf{B}_1\} - \mathbf{B}_1\Lambda_0 = \Lambda_1 - (\mathbf{A}_1 I)^T \tag{26}$$

$$A_0^T \mathcal{P}\{\mathbf{B}_2\} - \mathbf{B}_2\Lambda_0 = \Lambda_2 + \mathbf{B}_1\Lambda_1 - \mathbf{A}_1^T\mathbf{B}_1 - (\mathbf{A}_2 I)^T \tag{27}$$

etc. After substituting $\Lambda_0 = A_0^T$ in (26) and (27) we will have equations in the Hilbert space of square-integrable by measure $\mu(dy)$ $d \times d$-matrix-functions. By Fredholm alternative we at first have to test a normal solvability of these equations. One can prove that for that is sufficiently to integrate right parts by $\mu(dy)$ and to make sure that the results is zero. Then we should substitute $\Lambda_1 = \int_{\mathbb{Y}} \mathbf{A}_1(y)\mu(dy) := \bar{\mathbf{A}}_1$ in (26). Now we can find matrix $\mathbf{B}_1$, then substitute this matrix in (27), and test normal solvability of this equation. For that we need to have matrix Λ_2 in a following form:

$$\Lambda_2 = \overline{\mathbf{A}_1^T\mathbf{B}_1} - \mathbf{B}_1\bar{\mathbf{A}}_1^T + (\bar{\mathbf{A}}_2 I)^T$$

Then we can find matrix $\mathbf{B}_2$ and analyze further equation. As a result we will have expansions for matrices $\Lambda(\varepsilon)$ and $\mathbf{B}(\varepsilon)$ in working approximation of the first moment (23).

3 Second moments of linear Markov switched difference equations

To analyze second moments of solution (3) $x(t,s,x,y)$ we may write equation for matrices $x(t,s,x,y)(x(t,s,x,y))^T$

$$x_t(x_t)^T = A(y_t)x_{t-1}(x_{t-1})^T(A(y_t))^T \tag{28}$$

with initial condition $x_s(x_s)^T = xx^T$. If initial condition is given in a form $x_s = u(y_s)$ then covariance matrix is symmetric matrix-function $q_t(y_s) := u(y_s)(u(y_s))^T$. Therefore to derive formula for covariance matrices

$$q_t(y) = \mathbf{E}\{x(t,s,x,y_s)(x(t,s,x,y_s))^T/y_s = y\}, \ = s+1, s+2, ... \tag{29}$$

we may interpret equation (28) as linear Markov switched difference equation in $\mathbb{R}^{d^2}$ and apply methods and results of previous section. In this case instead of defined on the space $\mathbb{G}$ operator (8) we should deal with defined on the space of symmetric matrix-function $\mathbb{V} \subset \mathbb{R}^d \otimes \mathbb{R}^d$ operator $\mathbf{Q}$:

$$(\mathbf{Q}\mathbf{q})(y) = \int (A(z))^T q(z) A(z) p(y,dz) \tag{30}$$

Scalar product in $\mathbb{V}$ is defined by formula

$$< \mathbf{q}_1, \mathbf{q}_2 >:= \int\limits_{\mathbb{Y}} \text{Trace}\,[q_1(y)(q_2(y))]\,\mu(dy) \tag{31}$$

The same like in Lemma 1 we can prove that

$$\mathbf{q}_t(y) := \mathbf{E}\{X_s^{t+s} q(y_s)(X_s^{t+s})^T/x_s = y\} = [((\mathbf{Q}^t)\mathbf{q})(y))]^T \tag{32}$$

For equation (11) with near to constant matrix $A(y,\varepsilon)$ we also can apply derived in the second section algorithm and find working approximation of covariance matrices.

The major part of the third section is devoted to derived in [4] methods for mean square asymptotic analysis of equation (3). Not so difficult to prove that for any $q \in \mathbb{V}$ a behavior of $||\mathbf{Q}^t\mathbf{q}||$ as a function on t depends on spectral radius $r(\mathbf{Q}) := \max\{|\lambda| : \lambda \in \sigma(\mathbf{Q})$ of operator $\mathbf{Q}$. But spectral analysis of defined by formula (30) operator $\mathbf{Q}$ is very complicated problem even for finite phase space $\mathbf{Y}$. The proposal method and algorithm of mean square asymptotic analysis is based on well known Krein-Ruthman theory [14] of linear operators leaving as invariant reproducing cone in Banach space. We will order the space $\mathbb{V}$ by cone $\mathbb{K}$ of symmetric matrix-functions defined by formula:

$$\mathbb{K} := \{\mathbf{q} \in \mathbb{V} : \inf_{y \in \mathbb{Y}, ||x||=1} (q(y)x, x) \geq 0\} \tag{33}$$

with a set of inner points

$$\overset{\mathbf{o}}{\mathbb{K}} := \{\mathbf{q} \in \mathbb{V} : \inf_{y \in \mathbb{Y}, ||x||=1} (q(y)x, x) > 0\}. \tag{34}$$

This cone $\mathbb{K}$ permits to put space $\mathbb{V}$ in partial order using "inequality" $\mathbf{q}_1 << \mathbf{q}_2$ if $\mathbf{q}_2 - \mathbf{q}_1 \in \mathbb{K}$. If $\mathbb{Y}$ is compact it is easy to prove that $\mathbf{q} \in \overset{\mathbf{o}}{\mathbb{K}}$ if and only if there exists a such positive constant $c(q)$ that $\mathbf{q} >> c(q)I$ where I is the matrix unit of the space $\mathbb{V}$.

Theorem 1 *([4]) The next assertions are equivalent:*

(i) there exist such positive numbers $\rho < 1$ and M that

$$\mathbf{E}\{|x(t,s,x,y|^2\} \le M\rho^{t-s}|x|^2 \quad (35)$$

for all $x \in \mathbb{R}^d, y \in \mathbb{Y}$, and $t > s$;

(ii) there exists such $\mathbf{q} \in \overset{\mathbf{o}}{\mathbb{K}}$ that

$$\mathbf{Qq} - \mathbf{q} = -I; \quad (36)$$

(iii) maximal positive real spectrum point $r(\mathbf{Q})$ of operator $\mathbf{Q}$ is less than one.

Most productively we can apply this results to equation (3) switched by sequence i.i.d. random variables $\{y_t, t \in \mathbb{Z}\}$ with distribution $\mu(dy)$. In this case operator $\mathbf{Q}$ leaves as invariant the cone $\overset{\mathbf{o}}{\mathbb{K}}_d$ of constant positive defined symmetric $d \times d$-matrices and may be given for any $\mathbf{q} \in \overset{\mathbf{o}}{\mathbb{K}}_d$ by formula:

$$\mathbf{Qq} = \int (A(y))^T q A(y)\mu(dy) \quad (37)$$

Therefore equation (36) is an ordinary linear algebraic equation in space $\mathbb{R}^{\frac{d(d-1)}{2}}$. We can find symmetric matrix $\mathbf{q}$ from this equation and test for positive definition.

4 Stationary GARCH(p,q) process

The most popular mathematical model ([9],[17]) for the log of cumulative excess returns $Y_k := \ln S(t_k)$ is AR(1) regressive equation $Y_{k+1} = a + bY_k + \varepsilon_k$, where residuals ε_k are identically distributed bet have non-constant conditional variances $\sigma_k^2 := \mathbf{E}\{\varepsilon_{k+1}^2/\mathfrak{F}^k\}$ given by difference equation (GARCH(p,q) equation):

$$\sigma_t^2 = \theta_0 + \sum_{k=1}^{p} \varphi_k \sigma_{t-k}^2 + \sum_{k=1}^{q} \theta_k \xi_{t-k}^2, \quad (38)$$

$$\theta_0 > 0,\ \varphi_1 \ge 0, ..., \varphi_p \ge 0, \theta_1 \ge 0, ..., \theta_q \ge 0 \quad (39)$$

where $\{\xi_k\}$ is i.i.d. Gaussian $N(0,1)$ sequence. The coefficients of the regressive model (38) usually can be found by the least square method. For practical use of the above model a researcher should test [22] an existence of stationary stationary time series $\{\hat{\sigma}_t^2, t \in \mathbb{Z}\}$ which satisfy equation (38) and $\mathbf{E}\{\hat{\sigma}_t^4 < \infty$. Not so difficult to prove that the above assumption will be allowed if and only if the stationary solution of (38) is mean square asymptotically stable [18], that is, for any satisfying (38) time series $\{\sigma_t^2\}$ one may write

$$\lim_{t\to\infty} \mathbf{E}\{\left|\sigma_t^2 - \hat{\sigma}_t^2\right|^2\} = 0 \quad (40)$$

Not so difficult to prove that necessary condition for (40) is exponential decreasing of first moment $\mathbf{E}\{\sigma_t^2 - \hat{\sigma}_t^2\}$ and then we will use this assumption later. To avoid cumbersome calculations we will illustrate our proposal methods and results only for equation (38) with $p = 2, q = 2$. To analyze an existence of stationary solution we should deal with equation for difference $x_t := |\sigma_t^2 - \hat{\sigma}_t^2|^2$:

$$x_t = a_1 x_{t-1} + a_2 x_{t-2} + \sqrt{2}\theta_1 \eta_{t-1} x_{t-1} + \sqrt{2}\theta_2 \eta_{t-2} x_{t-2} \quad (41)$$

where $a_j = \theta_j + \varphi_j, j = 1, 2$, and $\eta_t = \frac{\xi_t^2 - 1}{\sqrt{2}}, t \in \mathbb{Z}$. The problem is: to find such a maximal region $\mathbf{S}(\theta_1, \theta_2, \varphi_1, \varphi_2) \subset \mathbb{R}^4$ that for any $\{\theta_1, \theta_2, \varphi_1, \varphi_2\} \in \mathbf{S}$ we can guarantee equality (40). For further analysis we put into equation (41) an artificial parameter r instead of multiplier $\sqrt{2}$:

$$x_t = a_1 x_{t-1} + a_2 x_{t-2} + r(\theta_1 \eta_{t-1} x_{t-1} + \theta_2 \eta_{t-2} x_{t-2}) \tag{42}$$

If $\theta_2 = 0$ we may apply the derived in the third section results rewrote equation (42) as linear difference equation for vector $\vec{x}_t := \{x_t, x_{t-1}\}^T$ in space $\mathbb{R}^2$:

$$\vec{x}_t = \begin{pmatrix} 0 & 1 \\ a_1 & a_2 \end{pmatrix} \vec{x}_{t-1} + r\eta_{t-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \vec{x}_{t-1}$$

and apply Theorem 1 for matrix $A(\eta_t)$ given by formula

$$A(\eta_t) = \begin{pmatrix} 0 & 1 \\ a_1 & a_2 \end{pmatrix} + r\eta_{t-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Now we can construct operator $\mathbf{A}_r$ for (42) in the space of symmetric matrix $\mathbf{q} = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}$ as a matrix in the space $\mathbb{R}^3$. According to Theorem 1 there exists such a positive number $\mathbf{r}$ that for any $r < \mathbf{r}$ the second moment of any solution of equation (42) tends to zero bet for any $r > \mathbf{r}$ there exists a solution with unbounded second moment. This boundary number one can find from equation $det\{\mathbf{A_r}\} = 0$, that is, from system of linear algebraic equation:

$$\begin{aligned} q_{12}(\phi_2 - 1) + (\phi_1 + \theta_1) q_{22} &= 0, \\ 2(\phi_1 + \theta_1)\phi_2 q_{12} - (1 - (\phi_1 + \theta_1)^2 - \phi_2^2 - \mathbf{r}^2 \theta_1^2) q_{22} &= 0. \end{aligned}$$

Equating to zero determinant of this system one can find a critical value $\mathbf{r}^2 = (1 + \phi_2)[(1 - \phi_2)^2 - (\phi_1 + \theta_1)^2]/\theta_1^2(1 - \phi_2)$. Therefore, GARCH(2,1) regressive model (41) have exponentially stable stationary variance with forth moment if and only if $r^2 = (\sqrt{2})^2 < \mathbf{r}^2$, that is:

$$\frac{(1 + \phi_2)[(1 - \phi_2)^2 - (\phi_1 + \theta_1)^2]}{\theta_1^2(1 - \phi_2)} > 2$$

More easy-to-use result we derive applying discrete Laplace transformation method [11] to second moments $R_t = \mathbf{E}\{|x_t|^2\}$ of satisfying (42) solution x_t. Let us assume that sequence $\{H_t, t \geq 1\}$ be solution of equation (42) with $r = 0$ on the assumptions of $H_0 = 1, \forall t \leq -1 : H_t = 0$ and $\{\tilde{x}_t, t > 0\}$ be solution of equation (42) with $r = 0$ on the assumptions of $\tilde{x}_{-1} = x_{-1}, \tilde{x}_0 = x_0$. Applying these notations we may rewrite equation (42) in a following form:

$$x_t = \tilde{x}_t + r \sum_{k=-1}^{t} H_{t-k-2}(\theta_1 \eta_{k+1} x_{k+1} + \theta_2 \eta_k x_k) \tag{43}$$

Raising to the second power and applying mathematical expectation we can obtain equation for $R_t = \mathbf{E}\{|x_t|^2\}$:

$$R_t = \tilde{R}_t + r^2 \sum_{k=0}^{t} (H_{t-k-1}\theta_1 + H_{t-k-2}\theta_2)^2 R_k \tag{44}$$

where $\tilde{R}_t = \mathbf{E}\{(\tilde{x}_t + rH_{t-1}\theta_1\eta_{-1}x_{-1})^2\}$. Now we can apply discrete Laplace transformation to (44) and write equation for $L_r(z) := \sum\limits_{t=0}^{\infty} R_t z^t$:

$$L_r(z) = \tilde{L}(z) + r^2 V(z) L_r(z) \tag{45}$$

where $\tilde{L}(z) := \sum\limits_{t=0}^{\infty} \tilde{R}_t z^t$ and $V(z) = \sum\limits_{t=0}^{\infty} (H_{t-1}\theta_1 + H_{t-2}\theta_2)^2 z^t$. According to Theorem 1 the necessary and sufficient condition for exponential decreasing of the second moments $R_t = \mathbf{E}\{|x_t|^2\}$ is convergence of series $\sum\limits_{t=0}^{\infty} R_t = L_r(1)$. This may be occure if and only if $r^2 < \mathbf{r}^2 := (V(1))^{-1}$. Using discrete Laplace transformation properties we can find the boundary $\mathbf{r}^{-2}$ in a form of contour integral. Finally we may guarantee an existance of the stationary GARCH(2,2) process with $\mathbf{E}\{\sigma_t^4 < \infty\}$ if and only if

$$2 < \mathbf{r}^2 = \left[\frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{z(\theta_1^2 + \theta_2^2) + (1 + z^2)\theta_1\theta_2}{(1 - (\theta_1 + \varphi_1)z - (\theta_2 + \varphi_2)z^2)(z^2 - (\theta_1 + \varphi_1)z - \theta_2 - \varphi_2)} dz \right]^{-1} \tag{46}$$

5 Asymptotic methods for Markov switched difference equations

The aim of this section is to propose an asymptotic methods for analysis of random iteration procedure in $\mathbb{R}^d$ given in a form of difference equation

$$x_{t+1} = x_t + \varepsilon f_1(x_t, y_t) + \varepsilon^2 f_2(x_t, y_t), \tag{47}$$

where right part depends on small positive parameter ε and ergodic homogeneous Feller Markov process y_t [7] on filtered probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}^t\}, \mathbf{P})$ with invariant measure $\mu(dy)$ and transition probability $p(y, dz)$ given on the set $\mathbb{Y} \subset \mathbb{R}$. We will assume that mappings $f_1 : \mathbb{R}^d \times \mathbb{Y} \to \mathbb{R}^d$ and $f_2 : \mathbb{R}^d \times \mathbb{Y} \to \mathbb{R}^d$ are continuous on $y \in \mathbb{Y}$, $f_1(x, y)$ has two bounded continuous x-derivatives $Df_1(x, y)$ and $D^2 f_1(x, y)$, and $f_2(x, y)$ has bounded continuous x-derivative $Df_2(x, y)$. Starting at $t = 0$ with given x_0, y_0 and applying iteration (47) one can generate vector $\{x_t, 0 \le t \le N\}$ for any N. But it is very complicated problem to find distribution of this vector for sufficiently large number N and therefore to find an approximation of the above distribution one can employ the limit theorems of contemporary probability theory (see [13], [20],[21] and references there). For that one can to construct the broken line in $\mathbb{R}^d$ with vertexes in the points $\{x_t\}$ by formula

$$s \in [t\varepsilon^2, (t+1)\varepsilon^2] : \; X_\varepsilon(s) = (x_{t+1} - x_t)(s\varepsilon^{-2} - t) + x_t \tag{48}$$

for all $t \in [0, N(\varepsilon^{-2})]$, where $N(\alpha)$ is integer part of number α. Applying limit theorem from [21] to distributions

$$\mathbf{P}_\varepsilon \sim \{X_\varepsilon(s), 0 \le s \le 1\}$$

we will construct limit distribution $\mathbf{P} \sim \{X(s), 0 \le s \le 1\}$ defined by stochastic Ito equation

$$dX(t) = a(X(s))ds + \sum_{k=1}^{d} \sigma_k(X(s))dW_k(s) \tag{49}$$

with initial condition $X(0) = x_0$, where vector-functions $a(x)$ and $\sigma_k(x)$, $k = 1, 2, ..., d$ are defined based on averaging by measure μ of functions $f_j(x, y)$, $j = 1, 2$ and its derivatives, and $\{W_k, k = 1, 2, ..., d\}$ are independent standard Wiener processes. The finite dimensional distribution of the solution of this equation $\{X(t\varepsilon^2), t = 0, 1, ..., N\}$ one can use to approximate distribution of solution of difference equation (47) $\{x_t, t = 0, 1, ..., N\}$ for any finite N. It should be mentioned that for analysis of (49) there are comprehensive facilities of contemporary stochastic analysis and mathematical physics. Besides we will prove that for sufficiently small ε to solve equilibrium asymptotical stability problem for (47) one can employ the second Lyapunov method derived for stochastic differential equations (49) in [13]. This problem has been discussed in many mathematical and engineering papers and the diffusion approximation approach at once has met with wide application in engineering and economical papers (see [1], [17] and references there). It should be mentioned that the above result has been developed for the analysis of equations on a finite time interval. But the averaging and diffusion approximation procedures in many applications we need to apply for asymptotic stability analysis of possible stationary solutions, that is, for analysis of differential equations as $t \to \infty$. Some of mentioned below results permissive to resolve this problem have been developed by author in [25] .
Let

$$(\mathcal{P}v)(y) := \int\limits_{\mathbb{Y}} v(z)p(y, dz) \tag{50}$$

be Markov operator defined on the space $\mathbb{C}(\mathbb{Y})$ of bounded continuous functions. We will assume that the spectrum $\sigma(\mathcal{P})$ has the simple eigenvalue 1, $\sigma(\mathcal{P})\diagdown\{1\} \subset \{z \in \mathbb{C} : |z| < \rho < 1\}$, and probability distribution $\{\mu(dy)\}$ is the solution of the equation $\mathcal{P}^*\mu = \mu$, where $\mathcal{P}^*$ is conjugate operator. Averaging procedure by the above invariant measure of any dependent on Markov process vector or matrix will be denoted with overline. Under these conditions one can extend [7] the potential of the above Markov process and to define the linear continuous operator by equality

$$(\Pi v)(y) := \sum_{k=0}^{\infty} (\mathcal{P}^k v)(y) \tag{51}$$

on the space $\bar{\mathbb{C}}(\mathbb{Y})$ of continuous functions $v \in \mathbb{C}(\mathbb{Y})$ with zero average $\bar{v} := \int\limits_{\mathbb{Y}} v(y)\mu(dy)$. This means that the equation $\mathcal{P}g - g = -v$ with $v \in \bar{\mathbb{C}}(\mathbb{Y})$ has unique solution (51) in $\bar{\mathbb{C}}(\mathbb{Y})$. Using the above Markov chain (47) we can define on the segment $[0, 1]$ step processes

$$s \in [t\varepsilon^2, (t+1)\varepsilon^2) : Y_\varepsilon(s) := y_t \tag{52}$$

If $\mathfrak{F}^t \subset \mathfrak{F}$, $t \geq 0$ is minimal filtration for stationary process y_t then for any $t \geq 0$ and $s \in [t\varepsilon^2, (t+1)\varepsilon^2)$ random vectors $X_\varepsilon(s)$ and $Y_\varepsilon(s)$ are $\mathfrak{F}^t$-measurable. To avoid cumbersome formulae we will denote conditional expectation $\mathbf{E}\{\xi/\mathfrak{F}^t\}|_{x_t=x, y_t=y}$ in abridged form $\mathbf{E}^t_{x,y}\{\xi\}$. To derive diffusion approximation for (47) we assume that $\bar{f}_1(x) \equiv 0$. Using the solution $x_t, t \in \mathbb{N}$ of difference equation (47) with initial condition $x_0 = x$ and Markov process y_t one can define the broken lines by formulae (48) and (52) for all $t \in [0, N(\varepsilon^{-2})]$. Under assumption that $\varepsilon \to 0$ one can apply the Skorokhod limit theorems from [21] for sequences of the above constructed processes and look for diffusion approximation of $\{X_\varepsilon(s), 0 \leq s \leq 1\}$ if the latter exists. To apply this theorem we should deal with operator family $\mathbf{L}(\varepsilon)$ defined for any sufficiently smooth function

$u(x, y)$ by equality

$$(\mathbf{L}(\varepsilon)u)(x,y) := \lim_{\delta\downarrow 0} \frac{1}{\delta}\mathbf{E}^t_{x,y}\{u(X^\varepsilon(s+\delta), Y^\varepsilon(s+\delta)) - u(X^\varepsilon(s), Y^\varepsilon(s))\} \tag{53}$$

If for any sufficiently smooth function $v(x)$ there exists such continuously dependent on ε function $v^\varepsilon(x, y)$ that $\lim\limits_{\varepsilon\to 0} v^\varepsilon(x, y) = v(x)$ and

$$\lim_{\varepsilon\to 0}(\mathbf{L}(\varepsilon)v^\varepsilon)(x,y) = (\mathbf{L}v)(x) \tag{54}$$

where $\mathbf{L}$ is diffusion operator

$$(\mathbf{L}v)(x) = \{(a(x), \triangledown) + (\sigma(x)\triangledown, \triangledown)\}v(x) \tag{55}$$

with positive defined symmetric matrix $\sigma(x)$, then one can approximate finite dimensional distributions of process $\{X_\varepsilon(t), 0 \le t \le 1\}$ by distributions of process $\{X(t), 0 \le t \le 1\}$ given as solution of stochastic differential equation (49) where symmetric matrices $\sigma_k(x), k = 1, 2, ..., d$ satisfy equation

$$\sum_{k=1}^{d} \sigma_k^2(x) = \sigma(x)$$

In our case, as it has been done in [25] for jump type Markov processes, for any arbitrary twice continuous differentiable on x function $v(x)$ we can look for the above mentioned function in a form of decomposition

$$v^\varepsilon(x,y) := v(x) + \varepsilon[((\Pi f_1)(x,y), \triangledown)v](x,y) + \varepsilon^2\hat{v}(x,y) \tag{56}$$

with bounded smooth function $\hat{v}(x, y)$. Now to derive formula (54) one has to present operator $\mathbf{L}(\varepsilon)$ accurate within $0(\varepsilon)$

$$\begin{aligned}\mathbf{L}(\varepsilon) = \tfrac{1}{\varepsilon^2}(\mathcal{P} - I) + \tfrac{1}{\varepsilon}(f_1(x,y), \triangledown)\mathcal{P} + \\ (f_2(x,y), \triangledown)\mathcal{P} + \tfrac{1}{2}(f_1(x,y), \triangledown)^2\mathcal{P} + 0(\varepsilon)\end{aligned} \tag{57}$$

to employ (57) to (56) and to decompose resulting function by powers of ε accurate within $0(\varepsilon)$:

$$\begin{aligned}(\mathbf{L}(\varepsilon)v^\varepsilon)(x,y) = \frac{1}{\varepsilon^2}(\mathcal{P} - I)v(x) + \\ \frac{1}{\varepsilon}\left[(f_1(x,y), \triangledown)v(x) + (\mathcal{P} - I)((\Pi f_1)(x,y), \triangledown)v(x)\right] + \\ (f_2(x,y), \triangledown)v(x) + \frac{1}{2}(f_1(x,y), \triangledown)^2 v(x)] + \\ (f_1(x,y), \triangledown)\mathcal{P}[(f_1(x,y), \triangledown)v(x)] + (\mathcal{P} - I)\hat{v}(x,y) + 0(\varepsilon)\end{aligned}$$

By this way using obvious equalities $(\mathcal{P} - I)\Pi = -I$, $(\mathcal{P} - I)v(x) \equiv 0$ and formula (54) we can write equation

$$\mathbf{L}v(x) = (f_2(x,y), \triangledown)v(x) + \frac{1}{2}(f_1(x,y), \triangledown)^2 v(x) \quad + \tag{58}$$

$$(f_1(x,y), \triangledown)[(\mathcal{P}\Pi f_1(x,y), \triangledown)v(x)] + (\mathcal{P} - I)\hat{v}(x,y) \tag{59}$$

and choose function $\hat{v}(x,y)$. By Fredholm alternative equation $(\mathcal{P}-I)\hat{v}(x,y)=g(x,y)$ has a solution if and only if $\bar{g}(x):=\int\limits_{\mathbb{Y}} g(x,y)\mu(dy)=0$. Therefore satisfying (58) function $\hat{v}(x,y)$ exists if and only if operator $\mathbf{L}$ is given by formula (55) with

$$\begin{aligned} a(x) &= \overline{f}_2(x)+\overline{[\mathcal{P}\Pi Df_1]^T f_1}(x) && (60)\\ \sigma(x) &= \frac{1}{2}[\overline{f_1 f_1^T}(x)+\overline{f_1\mathcal{P}\Pi f_1^T}(x)+\overline{(\mathcal{P}\Pi f_1)f_1^T}(x)] && (61)\end{aligned}$$

Remined that we have derived an approximate distribution of sequence $\{x_t, 0\le t\le N\}$ for any finite integer number N by distribution of solution of stochastic differential equation $\{X(s), 0\le s\le 1\}$. But some of application iterative procedures require asymptotic analysis of equation (47) as $t\to\infty$. For example discussing diffusion approximation approach to GARCH time series authors of papers [17] apply diffusion approximation for asymptotic stability analysis of stationary conditional variance. In our paper [5] we have proved legitimacy of diffusion approximation approach to time asymptotic of equation (47), assuming that point $x=0$ be an equilibrium, i.s. $f_1(0,y)\equiv 0$ and $f_2(0,y)\equiv 0$. The above method and algorithm we will illustrate later deriwing diffusion approximation for GARCH(1,1) process.
Let us remind of assumption $\bar{f}_1(x)\equiv 0$ which permits to apply diffusion approximation algorithm and to derive equation (55). If there exists such point x that $\bar{f}_1(x)\neq 0$ we may not divide segment $[0,1]$ by intervals of length ε^2 because Πf_1 does not exist and therefore there are singularity of order ε^{-1} in the definition of operator (57) as $\varepsilon\to 0$. To apply a diffusion approximation method in this case we at first should find solution of equation

$$\bar{x}_{t+1}=\bar{x}_t+\varepsilon\bar{f}_1(\bar{x}_t) \tag{62}$$

and to derive an asymptotic formula for so called *normalized deviations*

$$z_t := \frac{x_t-\bar{x}_t}{\sqrt{\varepsilon}} \tag{63}$$

Substituting $x_t=\sqrt{\varepsilon}z_t+\bar{x}_t$ in (47)

$$z_{t+1}=z_t+\sqrt{\varepsilon}g_1(\bar{x}_t,y_t)+\varepsilon[Df_1(\bar{x}_t,y_t)]z_t+o(\varepsilon), \tag{64}$$

where $g_1(x,y)=f_1(x,y)-\bar{f}_1(x)$, we nay apply to system (62)-(64) diffusion approximation approach. The sequence (63) gives rise to random processes

$$Z^\varepsilon(s)=\frac{X^\varepsilon(s)-\bar{X}^\varepsilon(s)}{\sqrt{\varepsilon}} \tag{65}$$

where $X^\varepsilon(s):=x_t, \bar{X}^\varepsilon(s):=\bar{x}_t$ for all $s\in[t\varepsilon,(t+1)\varepsilon)$ and any $t\in[0,N(\varepsilon))$. After substitution $Z^\varepsilon(s)$ instead of $X_\varepsilon(s)$, $[Df_1(\bar{X}(s),Y^\varepsilon(s))]Z^\varepsilon(s)$ instead of $f_2(X_\varepsilon(s),Y_\varepsilon(s))$ and $g_1(\bar{X}(s),Y^\varepsilon(s))$ instead of $f_1(X_\varepsilon(s),Y_\varepsilon(s))$ in corresponding formulae and vanishing $\sqrt{\varepsilon}$ one can approximate probability distribution $\mathbf{P}^Z_\varepsilon$ of process $Z^\varepsilon(s)$ by probability distribution $\mathbf{P}^Z$ of process Z satisfying stochastic differential equation equation

$$dZ(s)=D\bar{f}_1(\bar{X}(s))Z(s)ds+\sum_{k=1}^{d}\sigma_k(\bar{X}(s))dW_k(s) \tag{66}$$

where $\{W_k(s), k = 1, 2, ..., d\}$ are independent standard Wiener processes, and positive defined symmetric matrices $\{\sigma_k, r = 1, 2, ..., d\}$ have to be find from equality

$$\sum_{k=1}^{d} \sigma_k^2(x) = [\overline{g_1 g_1^T} + \overline{g_1 \mathcal{P}\Pi g_1^T} + \overline{(\mathcal{P}\Pi g_1) g_1^T}](x)$$

with initial condition $Z(0) = 0$. Deterministic function $\bar{X}(s)$ one can find as the solution of ordinary differential equation

$$d\bar{X}(s) = \bar{f}_1(\bar{X}(s))ds$$

with initial condition $\bar{X}(0) = x_0$. Roughly speaking for sufficiently small ε one can approximate distribution of the sequence $\{x_t, 0 \geq t \geq N(\varepsilon^{-1})\}$ by distribution of sequence $\{X(t\varepsilon) + \sqrt{\varepsilon}Z(t\varepsilon), 0 \geq t \geq N(\varepsilon^{-1})\}$.

6 Example

The most popular model [17] for the log of cumulative excess returns on a portfolio $p_t := \ln P_t$ is piecewise constant random process with uniform partition time $\{t_k, k = 1, 2, ...\}$ of the small length $t_{k+1} - t_k = h$ and jumps given by difference equation:

$$p_{t_{k+1}} = p_{t_k} + hc\sigma_k^2 + \sqrt{h}({}_h\xi_{t_{k+1}}) \tag{67}$$

where $\{{}_h\xi_t\}$ is non-correlated stationary time series with conditional variances $\sigma_{t_k}^2 = \mathbf{E}\{({}_h\xi_{t_{k+1}})^2/\mathfrak{F}^{t_k}\}$ given by GARCH(1,1) process:

$$\sigma_{t_{k+1}}^2 = \omega_h + \sigma_{t_k}^2\beta_h + h^{-1}\alpha_h({}_h\xi_{t_{k+1}}^2) \tag{68}$$

In the above equation h is small positive parameter and

$$1 - \alpha_h - \beta_h = h\theta + o(h), \omega_h = h\omega + o(h), \alpha_h = \frac{\sqrt{h}}{\sqrt{2}}\alpha + o(h)$$

Assuming that $\xi_{t_{k+1}} = {}_h z_{t_{k+1}}\sigma_{t_k}$ and market uncertainty $\{{}_h z_{t_k}, k \in \mathbb{Z}\}$ is Gaussian i.i.d. sequence with zero mean and variance h the author of paper [17] derives diffusion approximation for the above model in a form

$$dp = c\sigma^2 dt + \sigma dw_1(t), \tag{69}$$

$$d\sigma_t^2 = (\omega - \theta\sigma_t^2)dt + \alpha\sigma_t^2 dw_2(t) \tag{70}$$

where $w_1(t)$ and $w_2(t)$ are independent Wiener processes. It should be mentioned that the above result one can use only under assumption $\mathbf{E}\{z_{t_{k+1}} z_{t_k}\} = 0$. Our derived method of diffusion approximation permits to take into account possible correlation of market uncertainty $\{z_t\}$ that is very important for money portfolio analysis. To simplify further exposition we will write equations (67)-(68) as difference equation in $\mathbb{R}^2$

$$p_{t+1} = p_t + \varepsilon^2 c\sigma_t^2 + \varepsilon\sigma_t z_{t+1} \tag{71}$$

$$x_{t+1} = x_t + \varepsilon^2[\omega - \theta x_t] + \varepsilon\alpha y_t x_t \tag{72}$$

where $h = \varepsilon^2, x_t = \sigma_t^2, y_t = \frac{z_t^2-1}{\sqrt{2}}$, $\{z_t, t \in \mathbb{Z}\}$ is stationery Gaussian AR(1) process:

$$z_{t+1} = \rho z_t + \sqrt{1-\rho^2}\eta_{t+1}$$

and $\{\eta_t, t \in \mathbb{Z}\}$ is i.i.d. $N(0,1)$ series. By definition of potential Π and Markov operator $\mathcal{P}$ to find term $\overline{f_1 \mathcal{P} \Pi f_1^T}(x)$ in formula (60) we may [7] use decomposition of potential Π by powers of operator $\mathcal{P}$: $\Pi = (I - \mathcal{P})^{-1} = I + \sum_{k=1}^{\infty} \mathcal{P}^k$. Following our proposal method of diffusion approximation we should for equation (72) calculate diffusion coefficients (60) with $f_1(x,y) = \alpha y x$, $f_2(x,y) = \omega - \theta x$, that is, execute by operator $\mathcal{P}$ to function $g(y) = y$. Then for all $k \in \mathbb{N}$ we can write equation

$$\overline{(\mathcal{P}g)(\mathcal{P}^k g} = \int_{\mathfrak{Y}} \mathbf{E}\{y y_k / y_0 = y\} \mu(dy) = \rho^{2k} \tag{73}$$

The above assertions lead to following equations:

$$a(x) = \omega - \theta x + \alpha^2 x \sum_{l=1}^{\infty} \left\{ \int_{\mathbb{Y}} \mathbf{E}\{y_0 y_l\} \mu(dy) \right\} = \omega + \left[\alpha^2 \sum_{k=1}^{\infty} \rho^{2k} - \theta \right] x$$

$$\sigma^2(x) = \alpha^2 x^2 \int_{\mathbb{Y}} y^2 \mu(dy) + 2\alpha^2 x^2 \sum_{k=1}^{\infty} \rho^{2k} = \alpha^2 x^2 \left[1 + 2 \sum_{k=1}^{\infty} \rho^{2k} \right]$$

Therefore diffusion approximation for GARCH(1,1) process (70) has a form of Ito stochastic differential equation

$$d\sigma_t^2 = (\omega + (\alpha^2 \kappa - \theta)\sigma_t^2)dt + \alpha\sqrt{1+2\kappa}\sigma_t^2 dW(t) \tag{74}$$

where $\kappa = \sqrt{\frac{\rho^2}{1-\rho^2}}$. We have proved that this equation may be applied also for analysis of (70) as $t \to \infty$. According to [13] if

$$\alpha^2 \kappa - \theta - \frac{\alpha^2(1+2\kappa)}{2} = -\theta - \frac{\alpha^2}{2} < 0$$

there exists stationary solution $\hat{\sigma}_t^2$ of equation (74) and deviations $d_t := \sigma_t^2 - \hat{\sigma}_t^2$ of any other solution from this stationary process exponentially tend to zero as $t \to \infty$. In spite of the fact that process y_t in (72) has nonzero correlation this result no differs from similar result of the paper [17]. But to analyze distribution of satisfying (70) stationary process y_t we should take into account correlation coefficient ρ. Applying derived in [26] formula we can confirm that stationary solution of equation (74) has inverse Gamma-distribution, that is, random variables $\hat{\sigma}_t^{-2}$ have density function $f(x) = \frac{s^r x^{(r-1)}}{\Gamma(r)} e^{-sx}$, where $r = 1 + \frac{2(\theta - \alpha^2\kappa)}{\alpha^2(1+2\kappa)}, s = \frac{2\omega}{\alpha^2(1+2\kappa)}$.

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