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Operating with Fuzzy Probability Estimates in Decision Making Processes with Risk

Oleg Uzhga-Rebrov[1], Galina Kuleshova[2]
[1] *Rezekne Higher Education Institution*, [2] *Riga Technical University*

Abstract **– This paper considers different techniques of operating with fuzzy probability estimates of relevant random events in decision making tasks. The recalculation of posterior probabilities of states of nature based on the information provided by indicator events is performed using a fuzzy version of Bayes' theorem. The choice of an optimal decision is made on the basis of fuzzy expected value maximisation.**

Keywords **– Comparison of fuzzy numbers, fuzzy arithmetic, fuzzy expected value, restricted fuzzy arithmetic.**

I. Introduction

The theory of decision making under risk is a well-developed research and applied area. Effective tools aimed at modelling the initial situations as well as numerous choice criteria are developed that enable determination of optimal decisions for different systems of preferences of decision makers and different attitude to risk. The whole powerful apparatus successfully performs in situations when all factors of the task are set in the deterministic form.

The task of this paper is to represent techniques of decision making under risk in a fuzzy environment when all or some factors of the task are given in the fuzzy form.

II. Basics of Fuzzy Arithmetic

A fuzzy number is a fuzzy set specified in a set of real numbers. In practical tasks, triangular fuzzy numbers are frequently used. Some examples of this kind of numbers are shown in Fig. 1.

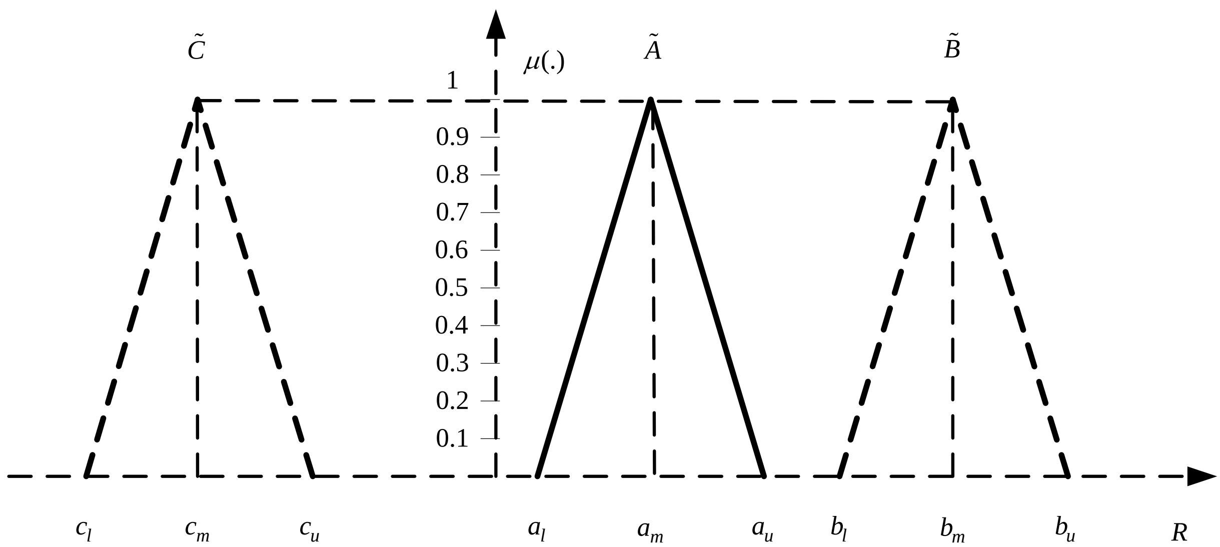

Fig. 1. Graphic representation of triangular fuzzy numbers in a set of real numbers R .

Intervals $[c_l, c_u]$, $[a_l, a_u]$, $[b_l, b_u]$, corresponding to values $\mu(\tilde{C})=0$, $\mu(\tilde{A})=0$ and $\mu(\tilde{B})=0$, are called supports of fuzzy numbers $\tilde{C}$, $\tilde{A}$, and $\tilde{B}$, respectively. The values c_m , a_m , b_m , to which the values of the membership functions $\mu(\tilde{C})=1$, $\mu(\tilde{A})=1$ and $\mu(\tilde{B})=1$ correspond, are called kernels of fuzzy numbers $\tilde{C}$, $\tilde{A}$, and $\tilde{B}$, respectively.

In general form, arithmetic operations on triangular fuzzy numbers are determined as follows [5], [7] (see Fig. 1):

$$-\tilde{A}=\left(-a_u-a_m-a_l\right) \tag{1}$$

$$\tilde{A}+\tilde{B}=\left(a_l+b_l, a_m+b_m, a_u+b_u\right). \tag{2}$$

$$\tilde{A}-\tilde{B}=\left(a_l-b_u, a_m-b_m, a_u-b_l\right). \tag{3}$$

$$\tilde{A}*\tilde{B}=\left(\min\left(a_lb_l, a_lb_u, a_ub_l, a_ub_u\right), a_mb_m, \max\left(a_lb_l, a_lb_u, a_ub_l, a_ub_u\right)\right) \tag{4}$$

$$\frac{\tilde{A}}{\tilde{B}} = \left(\min\left(\frac{a_l}{b_l}, \frac{a_l}{b_u}, \frac{a_u}{b_l}, \frac{a_u}{b_u} \right), \frac{a_m}{b_m}, \max\left(\frac{a_l}{b_l}, \frac{a_l}{b_u}, \frac{a_u}{b_l}, \frac{a_u}{b_u} \right) \right) \tag{5}$$

In decision making tasks, these approximate variants of multiplication and division operations on fuzzy numbers are commonly used [11]:

$$\tilde{A} * \tilde{B} \cong \left(a_l b_l, a_m b_m, a_u b_u \right) \tag{6, a}$$

if the supports of fuzzy numbers are in the range of positive real numbers (see Fig. 1).

$$\tilde{A} * \tilde{C} \cong \left(a_u c_l, a_m c_m, a_l c_u \right) \tag{6, b}$$

if the support of one fuzzy number is in the range of positive real numbers but the support of the other one is in the range of negative real numbers (see Fig. 1).

$$\frac{\tilde{A}}{\tilde{B}} \cong \left(\frac{a_l}{b_u}, \frac{a_m}{b_m} \frac{a_u}{b_l} \right). \tag{7}$$

In decision making tasks, the necessity to compare fuzzy numbers occurs. A great deal (>60) of relevant detailed methods are developed. These methods can be divided into three large classes.

(1) Methods using estimates of distances from centroids to certain original points. This kind of methods is described in [3] and [13].

(2) Methods using certain specific squares as an evaluation function. A method of this kind is proposed in [9], while a survey of such methods is provided in [6].

(3) Methods using the concept of maximal and minimal sets. An example of this kind of methods is given in [4].

Any of possible methods can be used to choose decisions in a fuzzy environment. But taking into account high computational complexity of most of the methods, the following simplified technique is commonly used. Let us have a look at Fig. 2, where two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are shown. It is necessary to determine which of these fuzzy numbers is greater. For that purpose, two estimated values $e_{\tilde{A}\tilde{B}}$ and $e_{\tilde{B}\tilde{A}}$ are calculated. The value of $e_{\tilde{B}\tilde{A}}$ is calculated as follows:

$$e_{\tilde{B}\tilde{A}} = \max_{p>r} \left\{ \min\left(\mu_{\tilde{A}}(p), \mu_{\tilde{B}}(r) \right) \right\}, p, r \in R \tag{8}$$

It is clear that $e_{\tilde{B}\tilde{A}} = 1$ for fuzzy numbers $\tilde{A}$ and $\tilde{B}$ in Fig. 2; that value is reached at point $r = 6$.

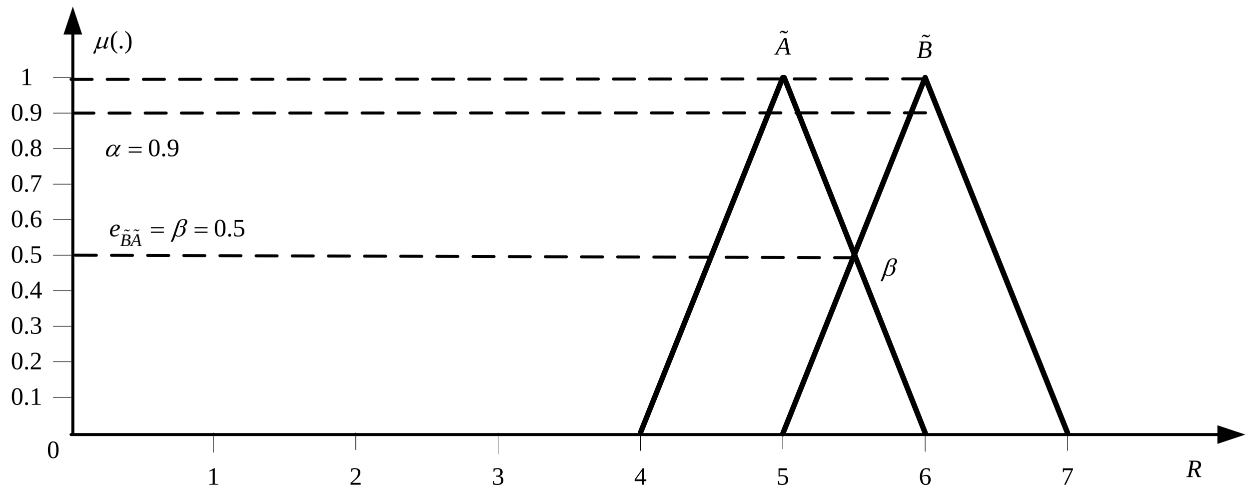

Fig. 2. Graphs of membership functions for the fuzzy numbers under comparison, $\tilde{A}$ and $\tilde{B}$.

The value $e_{\tilde{A}\tilde{B}}$ is determined as the value of the membership function for point β, in which graphs of membership functions $\mu(\tilde{A})$ and $\mu(\tilde{B})$ overlap. A fuzzy number $\tilde{B}$ is greater than a fuzzy number $\tilde{A}$ if

$$\tilde{B} > \tilde{A} \rightarrow e_{\tilde{B}\tilde{A}} = 1, e_{\tilde{A}\tilde{B}} \leq \alpha . \tag{9}$$

In different manuals on fuzzy decision analysis, the value α is calculated differently: $\alpha = 0.6$, $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$. It is apparent that the greater the value α is, the wider possibilities for comparing fuzzy numbers are. In this paper, the value $\alpha = 0.9$ will be used.

Since both conditions (9) fulfil for the fuzzy numbers $\tilde{A}$ and $\tilde{B}$, shown graphically in Fig. 2, an unambiguous conclusion can be made that $\tilde{B} > \tilde{A}$.

III. USING INFORMATION PROVIDED BY INDICATOR EVENTS

Let us assume that there is a complete group of random events (states of nature) which affect the outcomes of alternative decisions. Let there exist two states of nature, a_1 and a_2, and on the basis of expert estimation fuzzy values of probabilities of their occurrence are specified: $\tilde{p}(a_1)$ and $\tilde{p}(a_2)$. Let there be another complete group of random events (let us call them indicator events), whose occurrence probabilities depend on the states of nature. Let us assume that two indicator events exist, and fuzzy conditional probabilities $\tilde{p}(b_1/a_1)$, $\tilde{p}(b_1/a_2)$, $\tilde{p}(b_2/a_1)$ and $\tilde{p}(b_2/a_2)$ are evaluated. In partial case, the values of conditional probabilities can be set in a deterministic way.

Unfortunately, this kind of initial information cannot be employed directly in the process of decision making. The matter is that the decision maker is interested in the probabilities of occurrence of the states of nature when one or another indicator event happens. In other words, he is interested in the values of conditional probabilities $\tilde{p}(a_1/b_1)$, $\tilde{p}(a_2/b_1)$, $\tilde{p}(a_1/b_2)$ and $\tilde{p}(a_2/b_2)$. The calculation of the posterior conditional probabilities under consideration can be performed using a fuzzy version of Bayes' formula [2]:

$$\tilde{p}(a_i/b_j)=\left\{\frac{\tilde{p}(b_j/a_i)}{\sum_{i=1}^{n}\tilde{p}(b_j/a_i)}/S\right\}. \qquad (10)$$

Symbol S in (10) represents the requirement of allowability of deterministic values of probabilities $p(a_i/b_j)$ calculated on the supports of the corresponding fuzzy values of probabilities $\tilde{p}(a_i/b_j)$: the sum of those deterministic values has to be equal to 1. This requirement originates from the classic theory of probabilities; it is frequently a serious problem to meet the requirement when choosing specific values of probabilities $p(a_i/b_j)$. The problem can be sufficiently simplified if fuzzy probability estimates are consistent. In [12] the following definition of consistent fuzzy estimates is given: (1) supports of all corresponding fuzzy numbers are equal and symmetric with regard to their kernels; (2) the sum of values of probabilities corresponding to the kernels of fuzzy numbers is equal to 1. In what follows, only consistent fuzzy probability estimates will be considered.

Let us consider a simple example of calculation of fuzzy posterior values of probabilities.

Example 1. Let there be two random events (states of nature) a_1 and a_2. The prior fuzzy values of occurrence probabilities are set for those events: $\tilde{p}(a_1)=(0.50,0.60,0.70)$ and $\tilde{p}(a_2)=(0.30,0.40,0.50)$. There are two indicator events: b_1 and b_2 with the specified deterministic values of conditional probabilities: $p(b_1/a_1)=0.80$, $p(b_2/a_1)=0.20$, $p(b_1/a_2)=0.20$, $p(b_2/a_2)=0.80$. It is then necessary to calculate the values of the posterior probabilities $\tilde{p}(a_i/b_j) i,j=1.2$.

It is not difficult to make sure that fuzzy values of probabilities $\tilde{p}(a_1)$ and $\tilde{p}(a_2)$ are consistent; due to that, there is no need to define domains of possible values of the resulting fuzzy estimates.

For easier calculation, let us first calculate fuzzy complete values of probabilities of indicator event occurrence, $\tilde{p}(b_1)$, $\tilde{p}(b_2)$.

$$\tilde{p}(b_1)=p(b_1/a_1)*\tilde{p}(a_1)+p(b_1/a_2)*\tilde{p}(a_2)=0.80*\tilde{p}(a_1)+0.20*\tilde{p}(a_2)$$

$$\tilde{p}(b_2)=p(b_2/a_1)*\tilde{p}(a_1)+p(b_2/a_2)*\tilde{p}(a_2)=0.20*\tilde{p}(a_1)+0.80*\tilde{p}(a_2)$$

The calculations will be performed according to the rules of fuzzy arithmetic.

$$\tilde{p}(b_1)=0.80*(0.50,0.60,0.70)+0.20*(0.30,0.40,0.50)=$$

$$=(0.40,0.48,0.56)+(0.06,0.08,0.10)=(0.46,0.56,0.66)$$

$$\tilde{p}(b_2)=0.20*(0.50,0.60,0.70)+0.80*(0.30,0.40,0.50)=$$

$$=(0.10,0.12,0.14)+(0.24,0.32,0.40)=(0.34,0.44,0.54)$$

It is easy to see that fuzzy estimates $\tilde{p}(b_1)$ and $\tilde{p}(b_2)$ are consistent. Let us calculate fuzzy posterior values of probabilities $\tilde{p}(a_i/b_j) i,j=1.2$.

$$\tilde{p}(a_1/b_1)=\frac{p(b_1/a_1)*\tilde{p}(a_1)}{\tilde{p}(b_1)}=\frac{0.80*(0.50,0.60,0.70)}{(0.46,0.56,0.66)}=\frac{(0.40,0.48,0.56)}{(0.46,0.56,0.66)}=(0.848,0.857,0.870)$$

$$\tilde{p}(a_2/b_1)=\frac{p(b_2/a_1)*\tilde{p}(a_2)}{\tilde{p}(b_1)}=\frac{0.20*(0.30,0.40,0.50)}{(0.46,0.56,0.66)}=\frac{(0.06,0.08,0.10)}{(0.46,0.56,0.66)}=(0.130,0.143,0.152)$$

$$\tilde{p}(a_1/b_2)=\frac{p(b_2/a_1)*\tilde{p}(a_1)}{\tilde{p}(b_2)}=\frac{0.20*(0.50,0.60,0.70)}{(0.34,0.44,0.54)}=\frac{(0.10,0.12,0.14)}{(0.34,0.44,0.54)}=(0.259,0.273,0.294)$$

$$\tilde{p}(a_2/b_2)=\frac{p(b_2/a_2)*\tilde{p}(a_2)}{\tilde{p}(b_2)}=\frac{0.80*(0.30,0.40,0.50)}{(0.34,0.44,0.54)}=\frac{(0.24,0.32,0.40)}{(0.34,0.44,0.54)}=(0.706,0.727,0.741)$$

IV. CHOOSING DECISIONS IN FUZZY ENVIRONMENT

Let us consider solving that task using an example.

Example 2. There are two alternative decisions d_1 and d_2. Outcomes of the alternative decisions are influenced by indicator events b_1 and b_2 and states of nature a_1 and a_2. The values of fuzzy full probabilities of occurrence of indicator events and the values of fuzzy posterior probabilities of the states of nature are calculated in Example 1. Initial decision making situation is represented as a decision tree in Fig. 3.

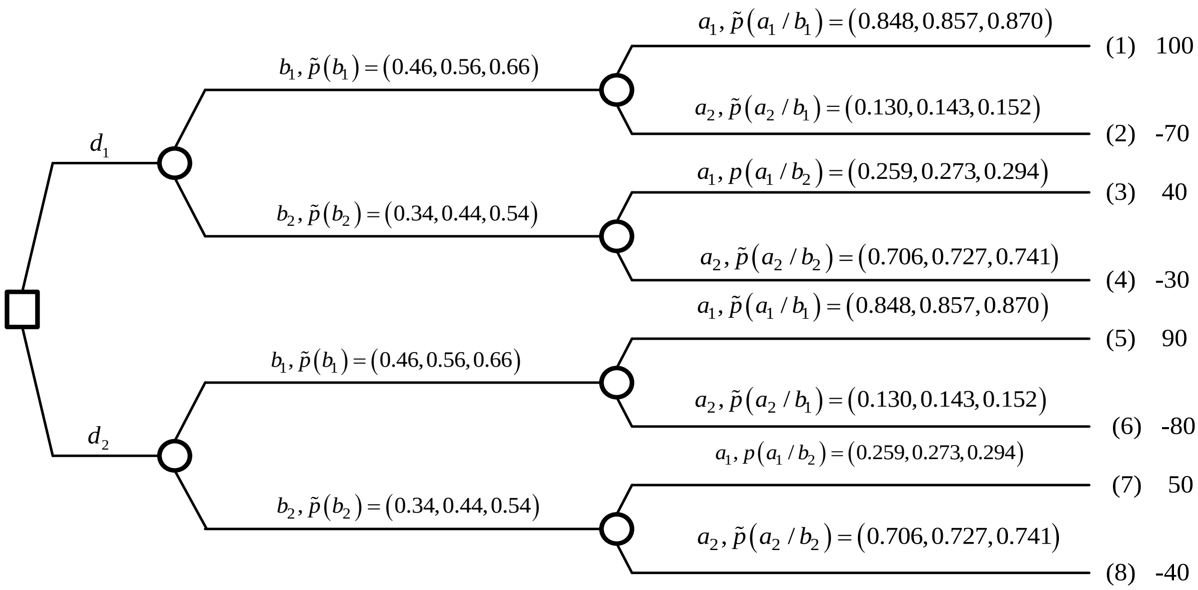

Fig. 3. A decision tree representing initial decision making situation in Example 3.

The numbers at the outcomes denote criteria estimates of the outcomes (in conditional monetary units). It is necessary to choose an optimal decision on the basis of the criterion of the expected value maximisation, which is calculated for each alternative decision over the whole set of outcomes.

Let us calculate fuzzy values of outcome probabilities. For that purpose, let us multiply the values of probabilities of random events leading to the given outcome.

$$p(1) = p(5) = \tilde{p}(b_1) * \tilde{p}(a_1/b_1) = (0.46, 0.56, 0.66) * (0.848, 0.857, 0.870) = (0.390, 0.480, 0.574)$$

$$p(2) = p(6) = \tilde{p}(b_1) * \tilde{p}(a_2/b_1) = (0.46, 0.56, 0.66) * (0.130, 0.143, 0.152) = (0.060, 0.080, 0.100)$$

$$p(3) = p(7) = \tilde{p}(b_2) * \tilde{p}(a_1/b_2) = (0.34, 0.44, 0.54) * (0.259, 0.273, 0.294) = (0.088, 0.120, 0.159)$$

$$p(4) = p(8) = \tilde{p}(b_2) * \tilde{p}(a_2/b_2) = (0.34, 0.44, 0.54) * (0.706, 0.727, 0.741) = (0.240, 0.320, 0.400)$$

Let us calculate fuzzy expected values for each alternative decision.

$$\tilde{K}(d_1) = 100 * (0.390, 0.480, 0.574) - 70 * (0.060, 0.080, 0.100) + 40 * (0.088, 0.120, 0.159) -$$

$$-30 * (0.240, 0.320, 0.400) = (39.00, 48.00, 57.40) + (-7.00, -5.60, -4.20) +$$

$$+(3.52, 4.80, 6.36) + (-12.00, -9.60, -7.20) = (23.52, 37.60, 52.36)$$

$$\tilde{K}(d_2) = 90 * (0.390, 0.480, 0.574) - 80 * (0.060, 0.80, 0.100) + 50 * (0.088, 0.120, 0.159) -$$

$$-40(0.240, 0.320, 0.400) = (35.10, 43.20, 51.66) + (-8.00, -6.40, 4.80) +$$

$$(4.40, 6.00, 7.95) + (-16.00, -12.00, -9.60) = (15.50, 30.00, 45.21)$$

To make the comparison of the calculated fuzzy expected values, let us represent them graphically (Fig. 4).

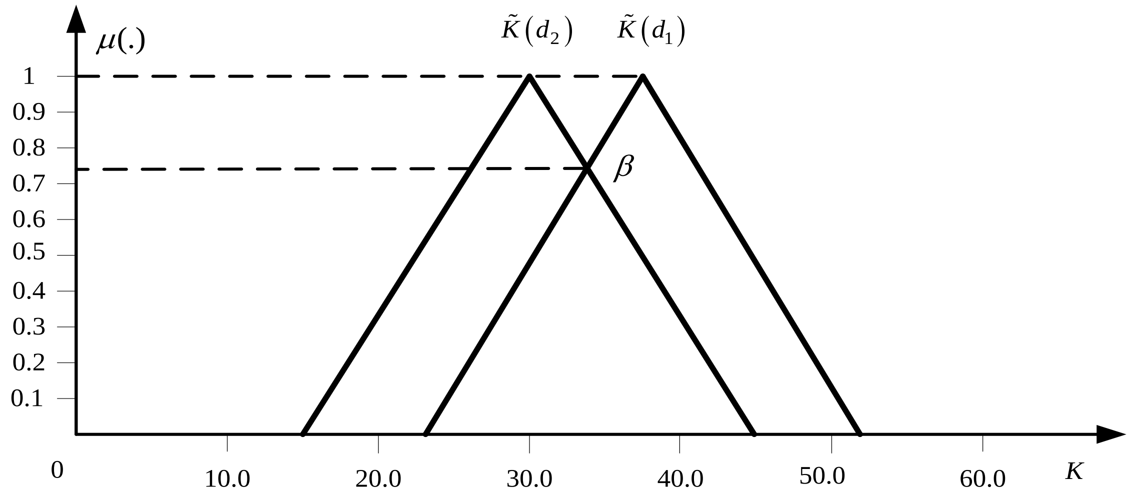

Fig. 4. Graphs of membership functions of fuzzy expected values $\tilde{K}(d_1)$ and $\tilde{K}(d_2)$ in Example 2.

As can be seen from Fig. 4, the value of membership function β for the overlapping point of the graphs is $\beta = 0.84$. It is apparent that both conditions (9) are satisfied for a fuzzy number $\tilde{K}(d_1)$. Due to that, $\tilde{K}(d_1) > \tilde{K}(d_2)$, and decision d_1 has to be chosen as optimal.

The result obtained is quite correct; however, it might cause mistrust of the decision maker because of the uncertainty of initial estimates and resulting expected values [1], [8], [10]. In this case, the following decision analysis can be made. Let us suppose that the decision maker is a risk-averse person. Then some probability values $p(2)$, $p(4)$, $p(6)$, $p(8)$ can be fixed that are closer to the right-hand borders of the supports of the corresponding fuzzy numbers. In other words, the question is about greater values of probabilities of unfavourable outcomes. However, the fixation of probability values has to be performed in such a way that those fixed values would be allowable values. In [12], formal algorithms for determining allowable values of probabilities are given for the cases of three and four fuzzy probability estimates. Though, it is possible to operate easier in the above example. Note that the sum of kernels of fuzzy probabilities of the outcomes is equal to 1. Then, to meet the requirement for the fixed values of probabilities, those values have to be such that conditions $p(1) + p(2) = 0.480 + 0.080 = 0.560$ and $p(3) + p(4) = 0.120 + 0.330 = 0.440$ are satisfied. The same conditions are valid for outcomes $p(5)$ and $p(6)$, $p(7)$ and $p(8)$. Then, at the fixed value of probability $p(2)$ the allowable value of probability $p(1)$ is calculated as $p(1) = 0.560 - p(2)$. Similarly, at the fixed value of probability $p(4)$ the allowable value of probability $p(3)$ is determined as $p(3) = 0.440 - p(4)$. Exactly the same relationships are valid for probabilities $p(5)$ and $p(6)$, $p(7)$ and $p(8)$.

Suppose that the decision maker has fixed these values of the probabilities of unfavourable outcomes: $p(2) = p(6) = 0.090$, $p(4) = p(8) = 0.370$. Then $p(1) = p(5) = 0.560 - 0.090 = 0.470$,
$p(3) = p(5) = 0.440 - 0.370 = 0.070$.

Let us calculate the deterministic expected values over outcomes of decisions d_1 and d_2 at the fixed values of the probabilities of the outcomes.

$$K(d_1) = 100 * 0.470 - 70 * 0.090 + 40 * 0.070 - 30 * 0.370 = 47.0 - 6.3 + 2.8 - 11.1 = 32.4$$

$$K(d_2) = 90 * 0.470 - 80 * 0.090 + 50 * 0.070 - 40 * 0.370 = 42.3 - 7.2 + 3.5 - 14.8 = 23.8$$

As an optimal decision, decision d_1 has to be chosen; this choice coincides with the choice made on the basis of fuzzy expected values.

If the result of choosing an optimal decision does not coincide with the one made on the basis of the fixed values of outcome probabilities, an additional analysis of decisions can be performed so as to validate one or another result of choice.

V. Conclusion

This paper has considered a technique for choosing decisions under risk in a fuzzy environment. Methods of fuzzy arithmetic enable successive execution of all operations related to the choice of an optimal decision. The use of a fuzzy version of Bayes' formula makes it possible to calculate fuzzy posterior probabilities of relevant random events (states of nature) on the basis of fuzzy information provided by indicator events. Fuzzy expected values are calculated over the whole set of consequences for each alternative decision. To compare resulting fuzzy estimates, any method designed for comparing fuzzy numbers can be used. The preference may be given to a simplified method represented in this paper because of its simplicity and visibility of results.

An analysis of the results obtained leads to these generalised conclusions:

1) Fuzzy initial probability estimates of relevant random events in decision making tasks are the consequence of the lack or insufficiency of reliable initial data.
2) Uncertainty of initial data is always translated into the uncertainty of the results.
3) Nowadays, an effective tool is available which helps to successfully solve decision making tasks in a fuzzy environment.

The fixation of values of fuzzy probabilities of consequences according to the rules of fuzzy arithmetic makes it possible to successfully perform an additional analysis of decisions, which is in essence a decision sensitivity analysis.

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Oleg Uzhga-Rebrov is a Professor at the Faculty of Economics, Rezekne Higher Education Institution (Latvia). He received his Doctor's degree in Information Systems from Riga Technical University in 1994. His research interests include different approaches to processing incomplete, uncertain and fuzzy information, in particular, fuzzy sets theory, rough set theory as well as fuzzy classification and fuzzy clustering techniques and their applications in bioinformatics.
Contact information: Rezekne Higher Education Institution, 90 Atbrivosanas aleja, Rezekne, LV-4600, Latvia.
E-mail: ushga@ru.lv

Galina Kuleshova is a Research Scientist at the Faculty of Computer Science and Information Technology, Riga Technical University (Latvia). She received her *M. Sc.* degree in Decision Support Systems from Riga Technical University. Current research interests include artificial neural networks, data mining, ontology engineering, classification methods and bioinformatics.
Contact information: Institute of Information Technology, Riga Technical University, 1 Kalku Str., Riga LV-1658, Latvia.
E-mail: galina.kulesova@cs.rtu.lv