DE GRUYTER OPEN
1862
RIGA TECHNICAL UNIVERSITY
Information Technology and Management Science
ISSN 2255-9094 (online)
ISSN 2255-9086 (print)
December 2017, vol. 20, pp. 91–96
doi: 10.1515/itms-2017-0016
https://www.degruyter.com/view/j/itms

Valuation of European Call Option via Inverse Fourier Transform

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Abstract **– Very few models allow expressing European call option price in closed form. Out of them, the famous Black–Scholes approach sets strong constraints – innovations should be normally distributed and independent. Availability of a corresponding characteristic function of log returns of underlying asset in analytical form allows pricing European call option by application of inverse Fourier transform. Characteristic function corresponds to Normal Inverse Gaussian (NIG) probability density function. NIG distribution is obtained based on assumption that time series of log returns follows APARCH process. Thus, volatility clustering and leptokurtic nature of log returns are taken into account. The Fast Fourier transform based on trapezoidal quadrature is numerically unstable if a standard cumulative probability function is used. To solve the problem, a dampened cumulative probability is introduced. As a computation tool Matlab framework is chosen because it contains many effective vectorization tools that greatly enhance code readability and maintenance. The characteristic function of Normal Inverse Gaussian distribution is taken and exercised with the chosen set of parameters. Finally, the call price dependence on strike price is obtained and rendered in XY plot. Valuation of European call option with analytical form of characteristic function allows further developing models with higher accuracy, as well as developing models for some exotic options.**

Keywords **– APARCH, European option, Fourier transform, normal inverse Gaussian distribution.**

I. Introduction

Black–Scholes model (BS) not only offers an elegant way for pricing derivatives but also imposes many restrictions. Thus, it is not possible to directly improve accuracy of calculations. However, the BS model can be used to develop a more sophisticated model; therefore, the fact that a characteristic function of the log returns can be directly calculated is employed. In the BS approach, there is an asset governed by Ito process [1], [2] and [3]:

$$S(t) = \mu S\mathrm{d}t + \sigma S\mathrm{d}W, \tag{1}$$

where μ – a drift rate;
σ – volatility.

The risk free asset with deterministic rate $r(t)$ also coexists. Bond prices are set by the following formula:

$$B(t) = \exp\int_0^t r\,(S)dS. \tag{2}$$

For numerical purposes, log return at maturity time is used:

$$X(T) = \log\left(\frac{S(T)}{S(0)}\right). \tag{3}$$

Random variable $X(T)$ is distributed according to true measure $\mathbb{P}$. There is also equivalent measure $\mathbb{Q}$, under which the discounted price will possess a martingale property. Under this risk neutral measure, the price of European call option follows:

$$P_{\text{call}} = B(T)\mathbb{E}_Q \max(S(T) - K, 0). \tag{4}$$

By restriction of BS $\mathbb{Q}$ is unique and $X(T)$ is normally distributed under both measures $\mathbb{P}$ and $\mathbb{Q}$ [1], [4], [5].

II. Nomenclature

log – natural logarithm;
S – spot price of underlying asset;
x – logarithmic spot price;
K – strike price of European option;
k – logarithmic strike price of European option;
T – maturity of European option.
APARCH – antisymmetric power autoregressive conditional heteroscedastic model;
BS – Black–Scholes model;
$\mathbb{P}$– probability measure;
$\mathbb{Q}$ – equivalent probability measure;
φ – characteristic function;
$\mathcal{F}$ – Fourier transformation;
PDF- probability density function;
$\mathcal{F}^{-1}$ – inverse Fourier transformation;
t – actual time;
τ – time to maturity ($T - t$);
X – random variable;
B – price of riskless bond;
$\mathbb{E}$ – expectation value operator;
W – standard Brownian motion;
u – variable in the dual space (after direct Fourier transformation).

III. Equations

A. Call Price Calculation by Fourier Transforms

In the following calculations, the anti-symmetric form of Fourier transformation will be used due to a reason that it is implemented in Matlab software package. Thus, multiplicator in front of integral will be missing:

$$\mathcal{F}[f](u) = \varphi(u) = \int_{\mathbb{R}} \exp(iux) f(x) dx. \quad (5)$$

However, inverse form of Fourier transformation will be without square root in front of integral:

$$\mathcal{F}^{-1}[\varphi](x) = f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-iux)\varphi(u) du.$$

Logarithmic transformation of strike price and spot price is introduced in the following way:

$$k = \log(K),$$
$$x = \log(S).$$

For every probability density function, there is a dual function, which uniquely depicts probability distribution. This function is called a characteristic function and is obtained by direct Fourier transformation of random variable [1]:

$$\varphi(t,u) = \mathbb{E}\left[\exp\left(iuX(t)\right)\right]. \quad (6)$$

The main assumption behind is that a characteristic function of log returns is available in analytical form. It is possible to completely recover PDF from a characteristic function

$$f(t,x) = \frac{1}{\pi} \int_0^\infty Re[\exp(-iux) f(t,u)] \mathrm{d}u. \quad (7)$$

Cumulative density function is then obtained in the following way:

$$F(t,x) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty Re\left[\frac{\exp(-iux)\varphi(t,u)}{iu}\right] \mathrm{d}u, \quad (8)$$

but the calculation of corresponding integral is numerically unstable; therefore, it is necessary to introduce the transformation to avoid numerical difficulties. As a result, "dampened" cumulative density has been introduced.

Dampened cumulative probability:

$$F^{\eta}(t,x) = \exp(-\eta k) P[X(t) \leq x]. \quad (9)$$

Dampened price for call option

$$P^{\eta}{}_{CALL}(k) = \exp(-\eta k) P_{\mathrm{CALL}}(k). \quad (10)$$

Fourier transform of the modified call

$$\psi^{\eta}(t,u) = \mathcal{F}[P^{\eta}{}_{\mathrm{CALL}}](u). \quad (11)$$

By substituting riskless bound, we obtain

$$\psi^{\eta}(T,u) = B(T) \int_{\mathbb{R}} \exp(iuk) P^{\eta}{}_{\mathrm{CALL}}(\mathrm{k}) \mathrm{d}k =$$
$$= \int_{\mathbb{R}} \exp(iuk) \left[\exp(\eta \mathrm{k}) \int_k^\infty \left(\exp(x) - \exp(k)\right) f(T,k) \mathrm{d}x\right] =$$
$$= \frac{B(T)}{(iu+\eta)(iu+\eta+1)} \varphi\left(T, u - i(\eta+1)\right), \quad (12)$$

where $f(T,k)$ – Normal inverse Gaussian probability function (see *B. Normal Inverse Gaussian Distribution*) [1].

The equation for pricing European call option using inverse Fourier transformation operator:

$$P_{\mathrm{CALL}}(k) = \exp(-k\eta) \mathcal{F}^{-1}[\psi(T,u;\eta)](k). \quad (13)$$

The equation for pricing European call option where Fourier transformation operator is expressed in analytical form [1]:

$$P_{\mathrm{CALL}}(k) = \frac{e^{-k\eta}}{\pi} \int_0^\infty Re[\exp(-iuk)\psi^{\eta}(T,u)] \, du.$$

B. Normal Inverse Gaussian Distribution

Probability density function

$$f(x) = \frac{\alpha\delta K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\pi\sqrt{\delta^2+(x-\mu)^2}} \mathrm{e}^{\delta\gamma+\beta(x-\mu)}, \quad (14)$$

where

K_1 – the modified Bessel function of third order and index 1;
μ – location parameter;
α – tail heaviness parameter;
β – asymmetry parameter;
δ – scale parameter;
$\gamma = \sqrt{\alpha^2 - \beta^2}$.

Using the corresponding characteristic function [9], it is possible to obtain:

$$\varphi(z) = \mathrm{e}^{i\mu z + \delta\left(\gamma - \sqrt{\alpha^2-(\beta+iz)^2}\right)}. \quad (15)$$

C. Time Series Analysis from Simple Models to a Specified ARCH Model

To obtain the corresponding density function (normal inverse Gaussian), the historical evolution of time-series is performed.

APARCH time-series approach is obtained by performing an analysis using standard time-series constructs augmented with additional elements unless acceptable accuracy is reached (see Fig. 1).

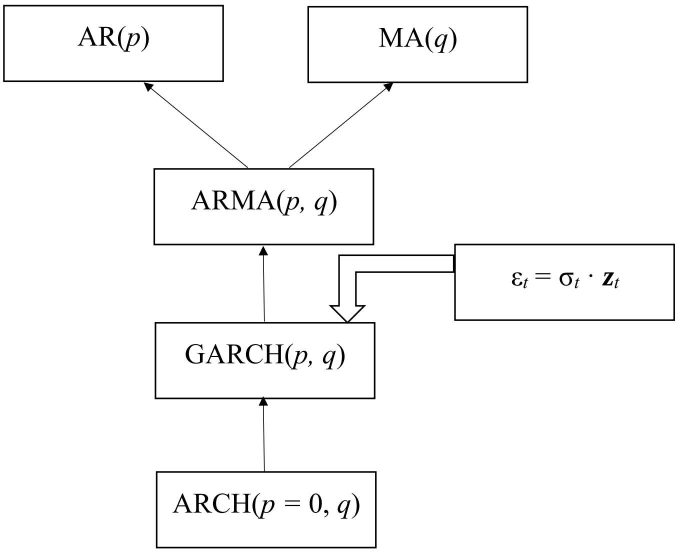

Fig. 1. Block scheme of APARCH time-series model [8].

$$\text{Let } r_t = \log\frac{S_t}{S_{t-1}} - \mathbb{E}\log\frac{S_t}{S_{t-1}}. \quad (16)$$

Let us introduce the following equation:

$$r_t = \mu_t + \varepsilon_t; \quad \varepsilon_t \sim N(0, \sigma_t^2), \quad (17)$$

where μ_t – average term;
ε_t – error term.

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \mathrm{A}_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \mathrm{B}_j \sigma_{t-j}^2, \quad (18)$$

where ω – constant;
A_i, B_i – GARCH model parameters.

If $q = 0$, this is the ARCH(p) volatility process [10].

D. APARCH Model

It is possible to show that

$$\sigma_t^2 = \omega + \mathrm{B}\sigma_{t-1}^2 + \mathrm{A}\sigma_{t-1}^2 f(z_{t-1}), \quad (19)$$

where $f(z_{t-1})$ is an innovation function [10], [6].

The most popular innovation function of GARCH models

$$f(z_{t-1}) = \\ = \begin{cases} z_{t-1}^2 \text{ Simple;} \\ (z_{t-1} - \theta)^2 \text{ Leverage;} \\ |z_{t-1} - \theta| - k(z_{t-1} - \theta)^2 \text{ News;} \\ (z_{t-1} - \theta)^{2\gamma} \text{ Power;} \\ |z_{t-1} - \theta| - \kappa(z_{t-1} - \theta)^{2\gamma} \text{ News and power,} \end{cases} \quad (20)$$

where θ – shifts the innovation function;
κ – news parameter tilts the innovation;
γ and ψ – flatten or steepen innovation function.

The task of the innovation function is to model asymmetry and news impact [10].

The GARCH models can be generalised by means of Box–Cox transformation:

$$\sigma_t^{\psi} = \omega + \mathrm{B}\sigma_{t-1}^{\psi} + \mathrm{A}\sigma_{t-1}^{\psi} f(z_{t-1}), \quad \text{with } f(z_{t-1}) = \\ = (z_{t-1} - \theta)^{2\psi}, \quad (21)$$

where θ – shifts the innovation function;
κ – news parameter tilts the innovation;
γ and ψ – flatten or steepen innovation function;
ω, $\mathrm{A}_i, \mathrm{B}_i$ – the remaining GARCH model parameters.

The APARCH (m, n) is written in the following way [10]:

$$X_t = \varepsilon_t, \; \varepsilon_t = \sigma_t z_t; \quad z_t \sim i.i.\mathrm{d}(0,1) \quad (22)$$

$$\sigma_t^{\Delta} = \omega + \sum_{i=1}^{m} \mathrm{A}_i (|\varepsilon_{t-i}| - \gamma_i \varepsilon_{t-i})^{\Delta} + \sum_{j=1}^{n} \mathrm{B}_j \sigma_{t-j}^{\Delta}, \quad (23)$$

with constraints $\omega > 0$; $\Delta \geq 0$; $\mathrm{A}_i \geq 0$; $-1 < \gamma < 1$, for $i = 1, \dots,\; m$, $\mathrm{B}_j \geq 0$, for $j = 1, \dots, n$ and

$$\sum_i^m k_i + \sum_j^n \mathrm{A}_j < 1, \text{where } k_i = \alpha \mathrm{A}_i (|\varepsilon_{t-i}| - \gamma \varepsilon_{t-i})^{\Delta}. \quad (24)$$

E. Generalised Hyperbolic Distribution

Definition (Generalised Hyperbolic distribution)

$$f_{\mathrm{GH}}(x;\, \alpha, \beta, \delta, \mu, \lambda) = \frac{(\gamma/\delta)^{\lambda}}{\sqrt{2\pi} K_{\lambda}} \frac{K_{\lambda - \frac{1}{2}}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\left(\sqrt{\delta^2 + (x-\mu)^2}\big/\alpha\right)^{\frac{1}{2} - \lambda}} e^{\beta(x-\mu)},$$

where $\gamma^2 = \alpha^2 - \beta^2$ and K_{λ} is a modified Bessel function of the third kind, with the index λ

$$\text{if } X \sim \mathrm{GH}\left(-\tfrac{1}{2}, \alpha, \beta, \delta, \mu\right), \quad (25)$$

then it has normal inverse Gaussian distribution [10].

F. Emergence of APARCH from Historical Consideration

Stochastic basis $\left(\Omega, \mathcal{F}, (\mathfrak{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ is introduced. $\mathbb{P}$ is an original physical probability measure and $\mathfrak{F}_t$ represents the information flow governed by Brownian motion $B = (B_t)_{t\in[0,T]}$ and Levi process $L = (\mathcal{L}_t)_{t\in[0,T]}$. Stock price is adopted to natural filtration $\mathfrak{F}_t$[3]. Daily return is defined as follows:

$$X_t = \log\frac{S_t}{S_{t-1}}. \quad (26)$$

Using the considerations above, it is possible to prove that spot price dynamics of underlying asset can be well described with normal inverse Gaussian distribution. [7], [11], [12].

IV. Numerical Algorithm for Valuation of European Call Option

A. Description of Steps

$\boxed{*}$ – element by element vector multiplication.

Example:

$$\vec{\boldsymbol{a}} \boxed{*} \vec{\boldsymbol{b}} = (a_1, a_2, \ldots\ldots, a_n) \boxed{*} (b_1, b_2, \ldots\ldots, b_n) = = (a_1 b_1, a_2 b_2, \ldots\ldots, a_n b_n). \quad (27)$$

$\boxed{/}$ – element by element vector division.

Example:

$$\vec{\boldsymbol{a}} \boxed{/} \vec{\boldsymbol{b}} = (a_1, a_2, \ldots\ldots, a_n) \boxed{/} (b_1, b_2, \ldots\ldots, b_n) = = \left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots\ldots, \frac{a_n}{b_n}\right). \quad (28)$$

1. Input the step sizes Δu (for grid in the dual space) and Δx (for grid in the original space), as well as the number of integration points N (identical in both spaces). Make sure that they satisfy $\Delta x \Delta u = \frac{2\pi}{N}$. Input also the "dampening parameter" η for the modified call.

2. Construct the vectors containing grid nodes in the dual space

$$\vec{\boldsymbol{u}} = \{(j-1)\Delta u : j = 1, \ldots N\}. \quad (29)$$

and in the original space

$$\vec{\boldsymbol{x}} = \left\{\frac{N}{2}\Delta x + (m-1) : m = 1, \ldots, N\right\}. \quad (30)$$

3. Construct the Fourier transform of the modified call

$$\vec{\boldsymbol{\psi}} = \exp(-rT)\varphi\big(T, \vec{\boldsymbol{u}} - i(\eta+1)\big) \boxed{/} [(i\vec{\boldsymbol{u}} + \eta) \boxed{*} (i\vec{\boldsymbol{u}} + \eta + 1)]. \quad (31)$$

4. Compute the vector

$$\vec{\boldsymbol{z}} = \exp(-ix_1\vec{\boldsymbol{u}}) \boxed{*} \psi. \quad (32)$$

5. For the trapezoidal rule set $z_1 = \frac{z_1}{2}$ and $z_N = \frac{z_N}{2}$.

Trapezoidal rule (in Fig. 2):

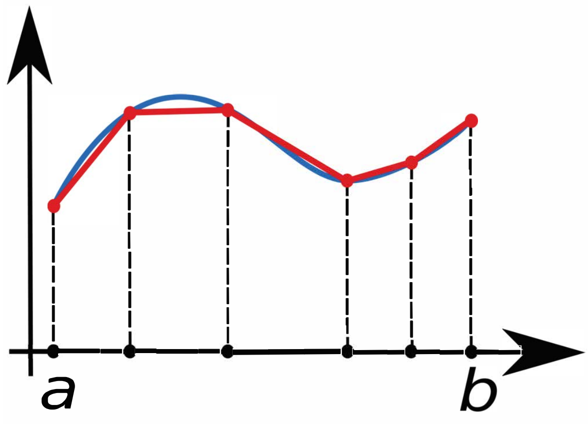

Fig. 2. Scheme of the iteration process (figure from [13]).

$$\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{n}\left(y_1 + y_2 + y_3 + \cdots + y_{n-1.} + \frac{y_0 + y_n}{2}\right).$$

6. Run the FFT on $\vec{\boldsymbol{z}} \rightarrow \vec{\boldsymbol{z}}^* = FFT(\vec{\boldsymbol{z}})$.

7. Compute option values

$$\vec{\boldsymbol{y}} = \frac{1}{\pi}\exp(-\eta\vec{\boldsymbol{x}}) \boxed{*} Re[\vec{\boldsymbol{z}}^*]. \quad (33)$$

8. Output the pair $(\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) \rightarrow$ the value y_j is a call option that corresponds to an option with log strike price x_j, for $j = 1, \ldots, N$ [1].

FFT is numerical routine that simultaneously calculates N sums

$$z_k^* = \sum_{j=1}^{N} \exp\left(-\frac{2\pi}{N}(j-1)(k-1)\right) z_j, \quad (34)$$

for $k = 1, \ldots, N$.

The order of numerical algorithm is $o(N\log N)$ [1]. The above-mentioned algorithm is illustrated in Fig. 3.

B. Flow Chart

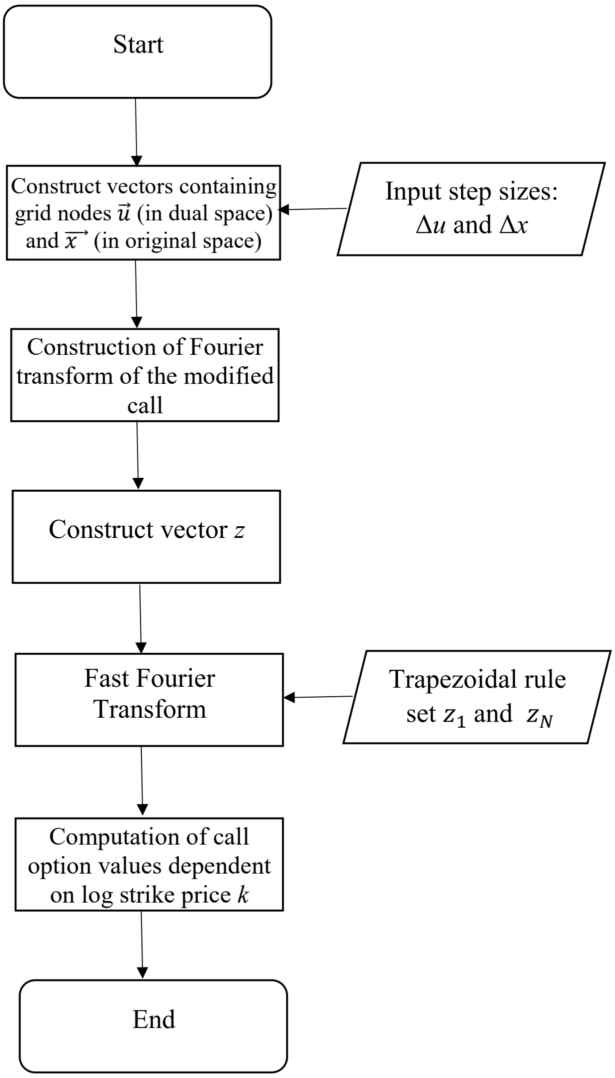

Fig. 3. Block scheme of the numerical algorithm.

If we fix the parameter k, then we can obtain call price dependence on time until maturity.

V. Results

A. Entry Parameters

The numerical simulation is performed with the following parameters:

```
characteristic function parameters
  pcf=
struct('t',10,'r',6.1,'delta',0.3,'alpha',
6,'beta',-4.52).
```

B. Output

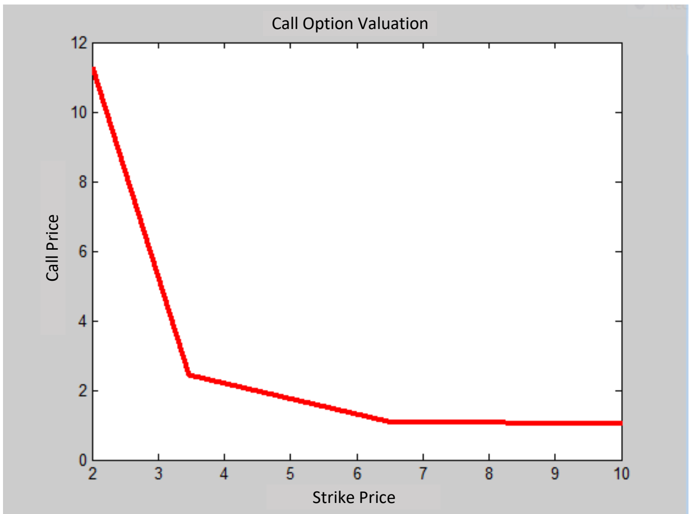

Fig. 4. The graph of the call price dependence.

Figure 4 shows call price dependence on strike price when maturity time is fixed with parametric dependence on NIG parameters, which describes historical dynamics of underlying asset. There is also explicit dependence on technical parameters (grid parameters).

Now it is possible to evaluate price of call option for a selling purpose. When sold, further hedging action should be introduced to avoid financial loses.

VI. Discussion

The developed Matlab code provides a quick valuation possibility of European call options. Next step is to increase accuracy of the algorithm by investigating and tuning distribution and technical parameters to obtain maximal accuracy. It could also be inevitable that further modification of distribution function itself is required. When accuracy requirements are met, it is possible to start developing a user-friendly interface. Besides, it would be very beneficial to apply the algorithm for cases of exotic options involved.

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