STOCHASTIC MODELING FOR AGE STRUCTURED POPULATION GROWTH UNDER ASSUMPTION OF SMALL FAST OSCILLATING PERTURBATIONS

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Abstract: We analyze Markov impulsive difference model for age structured population divided into three groups under the assumption of perturbations at random time moments. Assuming sufficiently small intervals between changes, we apply the stochastic averaging procedure and construct an ordinary 3-dimensional differential equation for population dynamics in the mean and a linear 3-dimensional stochastic differential equation for deviations from the mean trajectories. The results are applied for modelling of population dynamics using the data on Latvian residents collected by the Central Statistical Bureau of Latvia.

Keywords: averaged dynamics; stochastic approximation; Markov process.

Mathematics Subject Classification: Primary 60H30, 34K33; Secondary 34K07.

1 Introduction

Population aging is becoming an increasingly serious problem in most of developed countries. So not only demographers, but also economists, statisticians and mathematicians offer and analyze various models for population dynamics, applying recent methods. A wide description of such papers can be found in [1]. Samuelson used a simple demo-economic model for studying general questions about equilibrium in [6]. Many results were obtained for a static two-age group model, (see, for example, [2]). However, such type of a model cannot describe well the human economic life circle. As it was shown by Samuelson, it is crucial to separate the long middle period for people spent in employment, i.e. producing more than consuming. Like [5], we divide a population into three groups. Many authors use simplistic demographic assumptions. For instance, the model of Hock and Weil ignores childhood and early-adult mortality at all, assuming that death occurs only among the retired. In our model mortality exists in each group, moreover, with different coefficients. Bommier and Lee ([1]) studied overlapping generations models with realistic demography. The models were improved having been turned into dynamical. However, they still suffered from the lack of uncertainty.

We assume that increments in the groups caused by mortality and transferring people from one group to another have small random perturbations. We do not pretend that our model is detailed enough but it is closer to reality. The model can be improved from the viewpoint of demography. It is not our goal. We aim to offer a new approach to analysis of this type of demographical models, namely after introducing small perturbations we apply the averaging procedure ([8],[9]), then rewrite the model in a form of stochastic differential equations of Poisson type, define the normalized deviations of the solutions from averaged processes and apply the diffusion approximation ([7]-[9]) to these deviations.

2 The model

We consider the following split of a population of people into groups:

Group 1 – people who haven't been working since birth (youth dependents),
Group 2 – people having worked at least one day, but not retired (workers),
Group 3 – retired or disabled people (elderly dependents).

The following notations are used: $x(t)$ – the number of people in Group 1 at time t, $y(t)$ - the number of people in Group 2 at time t, $z(t)$ - the number of people in Group 3 at time t.
The finite difference scheme for the dynamics of the size of groups over the period Δ in moments $t_n = n\Delta, n \in N$ may be defined as follows:

$$x(t_{n+1}) = x(t_n) + \Delta_{11}x(t_n) - \Delta_{12}x(t_n) - \Delta_{13}x(t_n), \tag{1}$$
$$y(t_{n+1}) = y(t_n) + \Delta_{13}x(t_n) - \Delta_{21}y(t_n) - \Delta_{22}y(t_n), \tag{2}$$
$$z(t_{n+1}) = z(t_n) + \Delta_{22}y(t_n) - \Delta_{31}z(t_n), \tag{3}$$

where $\Delta_{11}x(t)$ is the number of births in period $[t, t+\Delta)$,
$\Delta_{12}x(t)$ is the number of deceased people in Group 1 in period $[t, t+\Delta)$;
$\Delta_{13}x(t)$ is the number of people who started working first time in period $[t, t+\Delta)$;
$\Delta_{21}y(t)$ is the number of deceased people in Group 2 in period $[t, t+\Delta)$;
$\Delta_{22}y(t)$ is the number of people who left Group 2, but stayed alive in period $[t, t+\Delta)$;
$\Delta_{31}z(t)$ is the number of deceased people in Group 3 in period $[t, t+\Delta)$.

The terms in (1)-(2)-(3) on a sufficiently small interval $[t, t+\Delta)$ are usually taken proportional to x, y, z respectively and to the length of interval Δ with some coefficient h_{jk}.

However, in fact, these coefficients are influenced by permanent random perturbations. In our paper these perturbations are supposed to be small, frequent and happening in random moments.

Increments in the number of people in groups are given by equalities:

$$\Delta_{1k}x_\varepsilon(t) = \begin{cases} \bar{h}_{1k}x_\varepsilon(t)\Delta + \varepsilon h_{1k}(\xi_{1k})x_\varepsilon(t), & if\Delta \geq \tau_{1k}, \\ \bar{h}_{1k}x_\varepsilon(t)\Delta, & if\Delta < \tau_{1k}, \end{cases} \quad k = 1,2,3;$$

$$\Delta_{2k}y_\varepsilon(t) = \begin{cases} \bar{h}_{2k}y_\varepsilon(t)\Delta + \varepsilon h_{2k}(\xi_{2k})y_\varepsilon(t), & if\Delta \geq \tau_{2k}, \\ \bar{h}_{2k}y_\varepsilon(t)\Delta, & if\Delta < \tau_{2k}, \end{cases} \quad k = 1,2; \tag{4}$$
$$\Delta_{31}z_\varepsilon(t) = \begin{cases} \bar{h}_{31}z_\varepsilon(t)\Delta + \varepsilon h_{31}(\xi_{31})z_\varepsilon(t), & if\Delta \geq \tau_{31}, \\ \bar{h}_{31}z_\varepsilon(t)\Delta, & if\Delta < \tau_{31}, \end{cases}$$

where ε is a small positive parameter, ξ_{jk} are uniformly distributed on the interval (0; 1) random variables involved in the value of perturbation, and the time between perturbations is exponentially distributed with parameter $\varepsilon^{-1}\lambda_{jk}$:

$$\xi_{jk} \sim Uniform(0,1)\,; \qquad P\,(\tau_{jk} > \Delta) = exp\{-\varepsilon^{-1}\lambda_{jk}\Delta\}.$$

Values of perturbations are equal to zero on average and random variables ξ_{jk} are independent on each other:

$$E\{\,h_{jk}(\xi_{jk})\} \underset{jk}{\equiv} 0 \qquad \{\xi_{jk}, \tau_{jk}, j = 1,2,3\,; k = 1,2,3\} \sim independent \tag{5}$$

Remark 1. The choice of the model in form (4)-(5) is caused by the aim to keep the property of predictability, i.e. dependence of $x_\varepsilon(t+s), y_\varepsilon(t+s), z_\varepsilon(t+s)$ for all $s > 0$ and $t \geq 0$ only on $x_\varepsilon(t), y_\varepsilon(t), z_\varepsilon(t)$.

Remark 2. Assume that ξ_{jk} has arbitrary continuous distribution $F_{jk}(u), u \in U_{jk} \subset R$. Then for any function $v(u)$ the following equality is fulfilled:

$$E\{\,v(\xi_{jk})\} = \int_{U_{jk}} v(u)dF_{jk}(u) = \int_0^1 v(F_{jk}^{-1}(u))du$$

if the integral exists. Therefore, all perturbations in (4) can be considered as functions of uniformly distributed random variables.

3 Derivation of the averaged dynamics equations

According to the definition, process $\vec{x}_\varepsilon(t) = \begin{pmatrix} x_\varepsilon(t) \\ y_\varepsilon(t) \\ z_\varepsilon(t) \end{pmatrix}$ possesses the Markov property (see [3]), so all its probabilistic characteristics are defined by the generator

$$L(\varepsilon)v(x,y,z) := \lim_{\Delta\to 0}\frac{1}{\Delta}E\{v(x_\varepsilon(t+\Delta), y_\varepsilon(t+\Delta), z_\varepsilon(t+\Delta)) - v(x_\varepsilon(t), y_\varepsilon(t), z_\varepsilon(t))/\{t,x,y,z\}\} \tag{6}$$

where $\{t,x,y,z\} \Leftrightarrow \{x_\varepsilon(t) = x, y_\varepsilon(t) = y, z_\varepsilon(t) = z\}$,

and $v(x,y,z)$ is a smooth enough bounded function. In derivation of (6), the asymptotical equality

$$P(\tau_{jk} \leq \Delta, \tau_{lm} \leq \Delta / l \neq j \vee k \neq m) = O(\Delta^2)$$

can be used, which gives us an opportunity to rewrite formula (6) in the following form:

$$L(\varepsilon)v(x,y,z) = \lim_{\Delta\to 0}\frac{1}{\Delta}E\{v(x_\varepsilon(t+\Delta), y_\varepsilon(t+\Delta), z_\varepsilon(t+\Delta)) - v(x_\varepsilon(t), y_\varepsilon(t), z_\varepsilon(t))/\{t,x,y,z\}\}$$
$$= L(\varepsilon)_x v(x,y,z) + L_y(\varepsilon)v(x,y,z) + L_z(\varepsilon)v(x,y,z), \quad (7)$$

where

$$L_x(\varepsilon)v(x,y,z) = \lim_{\Delta\to 0}\frac{1}{\Delta}E\{v(x_\varepsilon(t+\Delta), y, z) - v(x_\varepsilon(t), y, z)/x_\varepsilon(t) = x\} =$$
$$= (\bar{h}_{11} - \bar{h}_{12} - \bar{h}_{13})x\frac{\partial}{\partial x}v(x,y,z) + \varepsilon^{-1}\lambda_{11}\int_0^1[v(x+\varepsilon h_{11}(u)x, y, z) - v(x,y,z)]du +$$
$$+\varepsilon^{-1}\lambda_{12}\int_0^1[v(x-\varepsilon h_{12}(u)x, y, z) - v(x,y,z)]du + \varepsilon^{-1}\lambda_{11}\int_0^1[v(x-\varepsilon h_{13}(u)x, y, z) - v(x,y,z)]du,$$
$$L_y(\varepsilon)v(x,y,z) = \lim_{\Delta\to 0}\frac{1}{\Delta}E\{v(x, y_\varepsilon(t+\Delta), z) - v(x, y_\varepsilon(t), z)/y_\varepsilon(t) = y\} =$$
$$= [\bar{h}_{13}x - (\bar{h}_{22} + \bar{h}_{23})y]\frac{\partial}{\partial y}v(x,y,z) + \varepsilon^{-1}\lambda_{13}\int_0^1[v(x, y+\varepsilon h_{13}(u)x, z) - v(x,y,z)]du +$$
$$+\varepsilon^{-1}\lambda_{21}\int_0^1[v(x, y-\varepsilon h_{21}(u)y, z) - v(x,y,z)]du + \varepsilon^{-1}\lambda_{22}\int_0^1[v(x-\varepsilon h_{22}(u)y, y, z) - v(x,y,z)]du,$$
$$L_z(\varepsilon)v(x,y,z) = \lim_{\Delta\to 0}\frac{1}{\Delta}E\{v(x, y, z_\varepsilon(t+\Delta)) - v(x, y, z_\varepsilon(t))/z_\varepsilon(t) = z\} =$$
$$= (\bar{h}_{22}y - \bar{h}_{31}z)\frac{\partial}{\partial z}v(x,y,z) + \varepsilon^{-1}\lambda_{22}\int_0^1[v(x, y, z+\varepsilon h_{22}(u)y) - v(x,y,z)]du +$$
$$+\varepsilon^{-1}\lambda_{21}\int_0^1[v(x, y, z-\varepsilon h_{31}(u)z) - v(x,y,z)]du. \quad (8)$$

Differential equations of averaged dynamics are defined by operator

$$Lv(x,y,z) := \lim_{\varepsilon\to 0}L(\varepsilon)v(x,y,z) = \lim_{\varepsilon\to 0}L(\varepsilon)_x v(x,y,z) + \lim_{\varepsilon\to 0}L_y(\varepsilon)v(x,y,z) + \lim_{\varepsilon\to 0}L_z(\varepsilon)v(x,y,z). \quad (9)$$

We find

$$\lim_{\varepsilon\to 0}L_x(\varepsilon)v(x,y,z) = L_x^0 v(x,y,z) = \left(\bar{h}_{11}y - \bar{h}_{12}x - \bar{h}_{13}x\right)\frac{\partial}{\partial x}v(x,y,z),$$
$$\lim_{\varepsilon\to 0}L_y(\varepsilon)v(x,y,z) := L_y^0 v(x,y,z) = \left(\bar{h}_{13}x - \bar{h}_{21}y - \bar{h}_{22}y\right)\frac{\partial}{\partial y}v(x,y,z),$$
$$\lim_{\varepsilon\to 0}L_z(\varepsilon)v(x,y,z) := L_z^0 v(x,y,z) = \left(\bar{h}_{22}y - \bar{h}_{31}z\right)\frac{\partial}{\partial z}v(x,y,z)$$

and substitute in (9):

$$Lv(x,y,z) := \lim_{\varepsilon\to 0}L(\varepsilon)v(x,y,z)$$
$$= \lim_{\varepsilon\to 0}L(\varepsilon)_x v(x,y,z) + \lim_{\varepsilon\to 0}L_y(\varepsilon)v(x,y,z) + \lim_{\varepsilon\to 0}L_z(\varepsilon)v(x,y,z) =$$
$$= (\overline{h}_{11}y - \overline{h}_{12}x - \overline{h}_{13}x)\frac{\partial}{\partial x}v(x,y,z) + (\overline{h}_{13}x - \overline{h}_{21}y - \bar{h}_{22}y)\frac{\partial}{\partial y}v(x,y,z) +$$
$$+\left(\overline{h}_{22}y - \overline{h}_{31}z\right)\frac{\partial}{\partial z}v(x,y,z). \quad (10)$$

Therefore, the system of equations of averaged dynamics has form

$$\begin{cases} \frac{dx}{dt} = (\bar{h}_{11} - \bar{h}_{12} - \bar{h}_{13})x, \\ \frac{dy}{dt} = \bar{h}_{13}x - (\bar{h}_{21} + \bar{h}_{22})y, \\ \frac{dz}{dt} = \bar{h}_{22}y - \bar{h}_{31}z. \end{cases} \quad (11)$$

The analogue of the law of large numbers is: $\forall T>0, P(\lim_{\varepsilon\to 0} \sup_{0\le t\le T} |\vec{x}_\varepsilon(t) - \vec{x}(t)| - 0) = 1.$

The solution of system (11) can be easily found:

$$x(t) = x(0)e^{(\bar{h}_{11}-\bar{h}_{12}-\bar{h}_{13})t},$$
$$y(t) = y(0)e^{-(\bar{h}_{21}+\bar{h}_{22})t} + \alpha x(0)\left(e^{(\bar{h}_{11}-\bar{h}_{12}-\bar{h}_{13})t} - e^{-(\bar{h}_{21}+\bar{h}_{22})t}\right),$$
$$z(t) = x(0)e^{-\bar{h}_{31}t}\frac{\alpha}{(\bar{h}_{11}-\bar{h}_{12}-\bar{h}_{13}+\bar{h}_{31})(-\bar{h}_{21}-\bar{h}_{22}+\bar{h}_{31})}\Big(\left(1 - e^{(-\bar{h}_{21}-\bar{h}_{22}+\bar{h}_{31})t}\right)(\bar{h}_{11} - \bar{h}_{12} - \bar{h}_{13}) + \left(1 - e^{(\bar{h}_{11}-\bar{h}_{12}-\bar{h}_{13}+\bar{h}_{31})t}\right)(\bar{h}_{21} + \bar{h}_{22}) + \bar{h}_{31}\left(e^{(\bar{h}_{11}-\bar{h}_{12}-\bar{h}_{13}+\bar{h}_{31})t} - e^{(-\bar{h}_{21}-\bar{h}_{22}+\bar{h}_{31})t}\right)\Big) + y(0)e^{-\bar{h}_{31}t}\bar{h}_{22}\frac{e^{(-\bar{h}_{21}-\bar{h}_{22}+\bar{h}_{31})t}-1}{-\bar{h}_{21}-\bar{h}_{22}+\bar{h}_{31}} + z(0)e^{-\bar{h}_{31}t}, \quad (12)$$

where $\alpha = \frac{\bar{h}_{13}}{\bar{h}_{11}-\bar{h}_{12}-\bar{h}_{13}+\bar{h}_{21}+\bar{h}_{22}}$.

These processes with several realizations of initial difference system (1)-(3), taking into account (4)-(5), are shown on Figure 1. The data about Latvian residents were used. (Time unit is 1/100 of a year.)

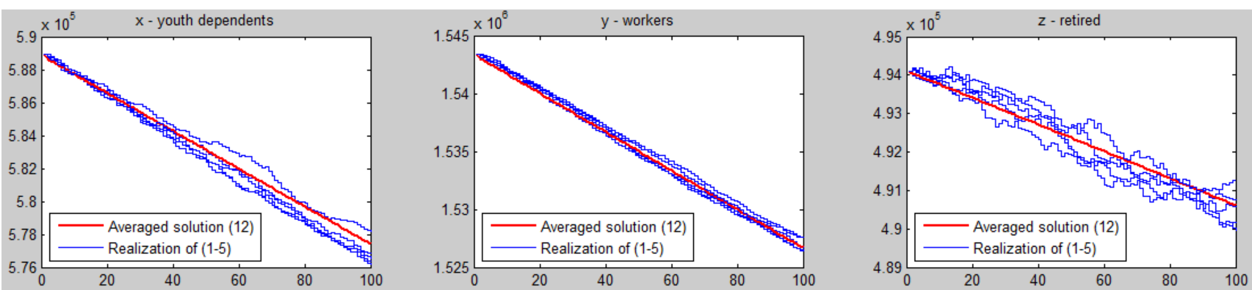

Fig.1

4 Approximate analysis of probabilistic characteristics of solutions

Markov process $\vec{x}_\varepsilon(t)$, which is the solution of (1)-(5), can be written as a system of stochastic differential equations of Poisson type:

$$
\begin{cases}
dx_\varepsilon(t) = (\bar h_{11} - \bar h_{12} - \bar h_{13})x_\varepsilon(t)dt + \varepsilon \int_0^1 h_{11}(u)x_\varepsilon(t)\mu_{11}(du,dt,\varepsilon) \\
\qquad -\varepsilon \sum_{k=2}^{3} \int_0^1 h_{1k}(u)x_\varepsilon(t)\mu_{1k}(du,dt,\varepsilon)\,, \\
dy_\varepsilon(t) = [\bar h_{13}x_\varepsilon(t) - (\bar h_{21} + \bar h_{22})y_\varepsilon(t)]dt + \varepsilon \int_0^1 h_{13}(u)x_\varepsilon(t)\mu_{13}(du,dt,\varepsilon) \\
\qquad -\varepsilon \sum_{k=1}^{2} \int_0^1 h_{2k}(u)y_\varepsilon(t)\mu_{2k}(du,dt,\varepsilon)\,, \\
dz_\varepsilon(t) = [\bar h_{22}y_\varepsilon(t) - \bar h_{31}z_\varepsilon(t)]dt + \varepsilon \int_0^1 h_{22}(u)y_\varepsilon(t)\mu_{22}(du,dt,\varepsilon) \\
\qquad -\varepsilon \int_0^1 h_{31}(u)z_\varepsilon(t)\mu_{31}(du,dt,\varepsilon)\,,
\end{cases}
\tag{13}
$$

where $\{\mu_{jk}(du,dt,\varepsilon), j = 1,2,3\,; k = 1,2,3\}$ are independent Poisson measures with parameters $\varepsilon^{-1}\lambda_{jk}dudt$,$u \in [0,1]$.

It is proved for sufficiently small ε that deviations of vector $\vec{x}_\varepsilon(t)$ from $\vec{x}(t)$ have the infinitesimal order $\sqrt{\varepsilon}$. Let us denote

$$
X_\varepsilon(t) = \frac{x_\varepsilon(t)-x(t)}{\sqrt{\varepsilon}}, \quad Y_\varepsilon(t) = \frac{y_\varepsilon(t)-y(t)}{\sqrt{\varepsilon}}, \quad Z_\varepsilon(t) = \frac{z_\varepsilon(t)-Z(t)}{\sqrt{\varepsilon}}, \quad \vec{X}_\varepsilon(t) = \begin{pmatrix} X_\varepsilon(t) \\ Y_\varepsilon(t) \\ Z_\varepsilon(t) \end{pmatrix}. \tag{14}
$$

According to the definition $\vec{X}_\varepsilon(t)$ is a non-homogeneous Markov process, and due to (11)-(13) its coordinates satisfy the system of stochastic equations of Poisson type:

$$
\begin{aligned}
dX_\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}}\left(dx_\varepsilon(t) - dx(t)\right) = \frac{1}{\sqrt{\varepsilon}}\Big[(\bar h_{11} - \bar h_{12} - \bar h_{13})\left(x(t) + \sqrt{\varepsilon}X_\varepsilon(t)\right)dt - \\
(\bar h_{11} - \bar h_{12} - \bar h_{13})x(t)dt\Big] + \sqrt{\varepsilon}\int_0^1 h_{11}(u)(x(t) + \sqrt{\varepsilon}X_\varepsilon(t))\mu_{11}(du,dt,\varepsilon) \\
- \sqrt{\varepsilon}\sum_{k=2}^{3}\int_0^1 h_{1k}(u)(x(t) + \sqrt{\varepsilon}X_\varepsilon(t))\mu_{1k}(du,dt,\varepsilon) = \\
= (\bar h_{11} - \bar h_{12} - \bar h_{13})X_\varepsilon(t))dt + \sqrt{\varepsilon}\int_0^1 h_{11}(u)x(t)\mu_{11}(du,dt,\varepsilon) \\
+ \varepsilon\int_0^1 h_{11}(u)X_\varepsilon(t)\mu_{11}(du,dt,\varepsilon) - \\
-\sqrt{\varepsilon}\sum_{k=2}^{3}\int_0^1 h_{1k}(u)x(t)\mu_{1k}(du,dt,\varepsilon) - \varepsilon\sum_{k=2}^{3}\int_0^1 h_{1k}(u)X_\varepsilon(t)\mu_{1k}(du,dt,\varepsilon),
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
dY_\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}}\left(dy_\varepsilon(t) - dy(t)\right) \\
= \frac{1}{\sqrt{\varepsilon}}\Big[\Big[\bar h_{13}\left(x(t) + \sqrt{\varepsilon}X_\varepsilon(t)\right) - (\bar h_{21} + \bar h_{22})\left(y(t) + \sqrt{\varepsilon}Y_\varepsilon(t)\right)\Big]dt \\
-[\bar h_{13}x(t) - (\bar h_{21} + \bar h_{22})y(t)]dt + \varepsilon\int_0^1 h_{13}(u)\left(x(t) + \sqrt{\varepsilon}X_\varepsilon(t)\right)\mu_{13}(du,dt,\varepsilon) - \\
\varepsilon\sum_{k=1}^{2}\int_0^1 h_{2k}(u)\left(y(t) + \sqrt{\varepsilon}Y_\varepsilon(t)\right)\mu_{2k}(du,dt,\varepsilon)\Big]
\end{aligned}
$$

$$= [\bar{h}_{13}X_{\varepsilon}(t)) - (\bar{h}_{21} + \bar{h}_{22})Y_{\varepsilon}(t)]dt + \sqrt{\varepsilon}\int_0^1 h_{13}(u)x(t)\mu_{13}(du, dt, \varepsilon) + \varepsilon\int_0^1 h_{13}(u)X_{\varepsilon}(t)\mu_{13}(du, dt, \varepsilon)$$

$$-\sqrt{\varepsilon}\sum_{k=1}^{2}\int_0^1 h_{2k}(u)y(t)\mu_{2k}(du, dt, \varepsilon) - \varepsilon\sum_{k=1}^{2}\int_0^1 h_{2k}(u)Y_{\varepsilon}(t)\mu_{2k}(du, dt, \varepsilon), \qquad (16)$$

$$dZ_{\varepsilon}(t) = \frac{1}{\sqrt{\varepsilon}}\big(dz_{\varepsilon}(t) - dz(t)\big)$$

$$= \frac{1}{\sqrt{\varepsilon}}\Big[\Big[\bar{h}_{22}\left(y(t) + \sqrt{\varepsilon}Y_{\varepsilon}(t)\right) - \bar{h}_{31}\left(z(t) + \sqrt{\varepsilon}Z_{\varepsilon}(t)\right)\Big]dt - \left[\bar{h}_{22}y(t) - \bar{h}_{31}z(t)\right]dt\Big] +$$

$$+\sqrt{\varepsilon}\int_0^1 h_{22}(u)\left(y(t) + \sqrt{\varepsilon}Y_{\varepsilon}(t)\right)\mu_{22}(du, dt, \varepsilon)$$

$$-\sqrt{\varepsilon}\int_0^1 h_{31}(u)\left(z(t) + \sqrt{\varepsilon}Z_{\varepsilon}(t)\right)\mu_{31}(du, dt, \varepsilon) =$$

$$= [\bar{h}_{22}Y_{\varepsilon}(t)) - \bar{h}_{31}Z_{\varepsilon}(t)]dt + \sqrt{\varepsilon}\int_0^1 h_{22}(u)y(t)\mu_{22}(du, dt, \varepsilon) + \varepsilon\int_0^1 h_{22}(u)Y_{\varepsilon}(t)\mu_{22}(du, dt, \varepsilon)$$

$$-\sqrt{\varepsilon}\int_0^1 h_{31}(u)z(t)\mu_{31}(du, dt, \varepsilon) - \varepsilon\int_0^1 h_{31}(u)Z_{\varepsilon}(t)\mu_{31}(du, dt, \varepsilon), \qquad (17)$$

where $\mu_{jk}(du, dt, \varepsilon)$ is the Poisson measure with parameter $\frac{1}{\varepsilon}\lambda_{jk}dudt, u \in [0,1]$.

The generator of non-homogeneous Markov process $\vec{X}_{\varepsilon}(t)$ is given by formula:

$$\mathscr{L}(\varepsilon)v(t, X, Y, Z) =$$

$$= \lim_{\Delta\to 0}\frac{1}{\Delta}E\{v(t+\Delta, X_{\varepsilon}(t+\Delta), Y_{\varepsilon}(t+\Delta), Z_{\varepsilon}(t+\Delta)) - v(t, X, Y, Z)/X_{\varepsilon}(t) = X, Y_{\varepsilon}(t)$$

$$= Y, Z_{\varepsilon}(t) = Z\} =$$

$$= \frac{\partial}{\partial t}v(t, X, Y, Z) + \mathscr{L}v(t, X, Y, Z) + O(\varepsilon),$$

where

$$\mathscr{L} = (\bar{h}_{11} - \bar{h}_{12} - \bar{h}_{13})X\frac{\partial}{\partial X} + [\bar{h}_{13}X - (\bar{h}_{21} + \bar{h}_{22})Y]\frac{\partial}{\partial Y} + (\bar{h}_{22}Y - \bar{h}_{31}Z)\frac{\partial}{\partial Z} -$$

$$-\lambda_{13}g_{13}x^2(t)\frac{\partial^2}{\partial X\partial Y} - \lambda_{22}g_{22}y^2(t)\frac{\partial^2}{\partial Y\partial Z} + \frac{1}{2}\sum_{k=1}^{3}\lambda_{1k}g_{1k}\,x^2(t)\frac{\partial^2}{\partial X^2} + \frac{1}{2}\lambda_{13}g_{13}x^2(t)\frac{\partial^2}{\partial Y^2} +$$

$$+\sum_{k=1}^{2}\lambda_{2k}g_{2k}y^2(t)\frac{\partial^2}{\partial Y^2} + \frac{1}{2}\big(\lambda_{22}g_{22}y^2(t) + \lambda_{31}g_{31}z^2(t)\big)\frac{\partial^2}{\partial Z^2}, \qquad (18)$$

$g_{jk} = \int_0^1 h_{jk}^2(u)du, \{j = 1,2,3; k = 1,2,3;\}$, and $v(t, X, Y, Z)$ is an arbitrary sufficiently smooth bounded function. One can write

$$\lim_{\varepsilon\to 0}\mathscr{L}(\varepsilon) = \frac{\partial}{\partial t} + \mathscr{L}. \qquad (19)$$

Therefore, we apply the method of diffusion approximation for approximate analysis of population dynamics, where only operator $\mathscr{L}$ is used, which is obtained in the result of taking the limit as $\varepsilon \to 0$. The limit operator has the following vector form:

$$\mathscr{L} = (A\vec{X}, \vec{\nabla}) + \frac{1}{2}(G(\vec{x}(t))\vec{\nabla}, \vec{\nabla}). \qquad (20)$$

We use the following notations here: $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, $\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \\ \frac{\partial}{\partial Z} \end{pmatrix}$,$\vec{X} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$,

$$G(\vec{x}) = \begin{pmatrix} (\lambda_{11}g_{11} + \lambda_{12}g_{12} + \lambda_{13}g_{13})x^2 & -\lambda_{13}g_{13}x^2 & 0 \\ -\lambda_{13}g_{13}x^2 & (\lambda_{21}g_{21} + \lambda_{22}g_{22})y^2 & -\lambda_{22}g_{22}y^2 \\ 0 & -\lambda_{22}g_{22}y^2 & \lambda_{31}g_{31}z^2 \end{pmatrix},$$

$$A = \begin{pmatrix} \bar{h}_{11} - \bar{h}_{12} - \bar{h}_{13} & 0 & 0 \\ \bar{h}_{13} & -\bar{h}_{21} - \bar{h}_{22} & 0 \\ 0 & \bar{h}_{22} & -\bar{h}_{31} \end{pmatrix}.$$

Formula (20) defines non-homogeneous Markov process $\{X(t), Y(t), Z(t)\}$. One can see that equation (15) for $X_\varepsilon(t)$ is not dependent on $Y_\varepsilon(t)$ and $Z_\varepsilon(t)$. So, it can be studied separately, using its diffusion approximation in a form of Ito stochastic differential equation:

$$dX(t) = (\bar{h}_{11} - \bar{h}_{12} - \bar{h}_{13})X(t)dt + \sqrt{\lambda_{11}g_{11} + \lambda_{12}g_{12} + \lambda_{13}g_{13}}\,x(t)dw(t). \qquad (21)$$

However, for the analysis of the three components $\{X_\varepsilon(t), Y_\varepsilon(t), Z_\varepsilon(t)\}$ of the model altogether it is necessary to use the diffusion approximation in the form of a system of Ito stochastic equations that has a more complicated form of the equation for $X(t)$:

$$\begin{cases} dX(t) = (\bar{h}_{11} - \bar{h}_{12} - \bar{h}_{13})X(t)dt + \sqrt{\lambda_{13}g_{13}}\,x(t)dw_1(t) + \sqrt{\lambda_{11}g_{11} + \lambda_{12}g_{12}}\,x(t)dw_2(t), \\ dY(t) = [\bar{h}_{13}X(t) - (\bar{h}_{21} + \bar{h}_{22})Y(t)]dt + \\ \qquad + \sqrt{\lambda_{13}g_{13}}\,x(t)dw_1(t) + y(t)\sqrt{\lambda_{21}g_{21}}\,dw_3(t) + y(t)\sqrt{\lambda_{22}g_{22}}\,dw_4(t), \\ dZ(t) = \bar{h}_{22}Y(t)dt - \bar{h}_{31}Z(t)dt + \\ \qquad + y(t)\sqrt{\lambda_{22}g_{22}}\,dw_4(t) + z(t)\sqrt{\lambda_{31}g_{31}}\,dw_5(t). \end{cases} \qquad (22)$$

where $\{w_k(t), k = 1,2,3,4,5\}$ are independent standard Wiener processes. By definition, the first moment of Gaussian Markov process $\{X(t), Y(t), Z(t)\}$ is equal to zero. The covariance matrix $\Sigma(t) = \begin{pmatrix} \sigma_{XX}(t) & \sigma_{XY}(t) & \sigma_{XZ}(t) \\ \sigma_{XY}(t) & \sigma_{YY}(t) & \sigma_{YZ}(t) \\ \sigma_{XZ}(t) & \sigma_{YZ}(t) & \sigma_{ZZ}(t) \end{pmatrix}$ of this process is given by matrix differential equation

$$\frac{d}{dt}\Sigma(t) = A\Sigma(t) + \Sigma(t)A^T + G(\vec{x}(t)) \qquad (23)$$

with zero initial conditions.

The equations for the elements of this matrix have form:

$$\frac{d}{dt}\sigma_{XX}(t) = 2(\bar{h}_{11} - \bar{h}_{12} - \bar{h}_{13})\sigma_{XX}(t) + (\lambda_{11}g_{11} + \lambda_{12}g_{12} + \lambda_{13}g_{13})x^2(t),$$

$$\frac{d}{dt}\sigma_{XY}(t) = (\bar{h}_{11} - \bar{h}_{12} - \bar{h}_{13})\sigma_{XY}(t) + \bar{h}_{13}\sigma_{XX}(t) - (\bar{h}_{21} + \bar{h}_{22})\sigma_{XY}(t) + \lambda_{13}g_{13}x^2(t),$$

$$\frac{d}{dt}\sigma_{XZ}(t) = 0,$$
$$\frac{d}{dt}\sigma_{YY}(t) = 2[\bar{h}_{13}\sigma_{XX}(t) - (\bar{h}_{21} + \bar{h}_{22})\sigma_{YY}(t)] + (\lambda_{21}g_{21} + \lambda_{22}g_{22})y^2(t),$$
$$\frac{d}{dt}\sigma_{YZ}(t) = \bar{h}_{13}\sigma_{XZ}(t) - (\bar{h}_{21} + \bar{h}_{22} - \bar{h}_{31})\sigma_{YZ}(t) + \bar{h}_{22}\sigma_{YY}(t) + \lambda_{22}g_{22}y^2(t),$$
$$\frac{d}{dt}\sigma_{ZZ}(t) = 2\bar{h}_{22}\sigma_{YZ}(t) - 2\bar{h}_{31}\sigma_{ZZ}(t) + \lambda_{31}g_{31}z^2(t) \tag{24}$$

For approximative analysis of probabilistic characteristics of the initial process $\vec{x}_\varepsilon(t) = \begin{pmatrix} x_\varepsilon(t) \\ y_\varepsilon(t) \\ z_\varepsilon(t) \end{pmatrix}$ one can use formulae

$$x_\varepsilon(t) \approx \tilde{X}(t) = x(t) + \sqrt{\varepsilon}X(t), y_\varepsilon(t) \approx \tilde{Y}(t) = y(t) + \sqrt{\varepsilon}Y(t), z_\varepsilon(t) \approx \tilde{Z}(t) = z(t) + \sqrt{\varepsilon}Z(t) \tag{25}$$

For small ε this approximation gives sufficiently good results.

Realizations of $x_\varepsilon(t)$ taking into account equation (21) and the averaged solution (12), together with the solution of the averaged equation are given on Figure 2.

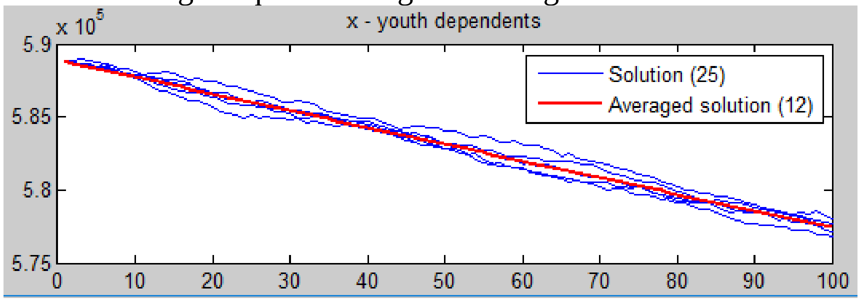

Fig.2

Realizations of all three approximative processes $x_\varepsilon(t)$, $y_\varepsilon(t)$ and $z_\varepsilon(t)$ obtained altogether, using (22) and the averaged solution (12), together with correspondent averaged processes are shown on Figure 3.

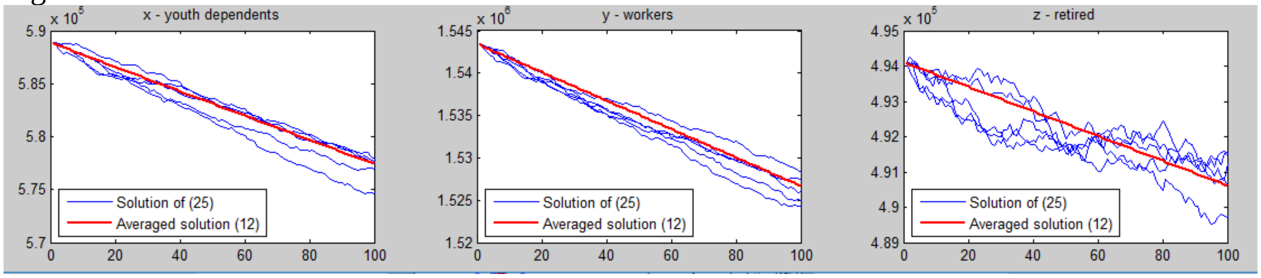

Fig.3

4 Conclusion

The value of our work is in applying of averaging and diffusion approximation to the considered demographical model. For the analyzed system of difference equations that describes age structured population growth, the averaged equations, as well as stochastic differential equations for the limit process of the normalized deviations of the solutions from the averaged solutions, are derived. The results are illustrated with data on Latvian residents.

Without any doubt, the obtained stochastic model is better than the correspondent deterministic model, however, it also can be improved further. The model is closed, i.e. it does not include emigration and immigration, which is a significant factor of the change of values in the first two groups, especially for such open countries as Latvia. Other demographic parameters can be included into the model as well.

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