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STOCHASTIC APPROXIMATION OF THE LOGISTIC SYSTEM DYNAMICS

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Abstract. The stochastic dynamic of logistic system problem addresses the movement of vehicles between their locations over a given planning horizon. This problem deals with a logistic system consisting of a wholesale store, a retail store and automobiles which are taking part in goods delivery from a wholesale store to a retail store. Assuming the demands are random and coming at random time moments, we construct a stochastic model for this transport logistic scheme and derive Gaussian approximation for transport and stock level of goods dynamics.

Keywords: stochastic approximation, transport logistic system, Gaussian approximation.

Mathematics Subject Classification: Primary 92B05, 60H30; Secondary 60G57, 37H10

1 Introduction

In the last years, research interest in transport logistics has increasingly focused on dynamical theory approach (see [5,6,7,8] and references there) for quantitative and qualitative analysis of system behavior. The stochastic dynamic of logistic system problem arises when a carrier must allocate vehicles over space and time in an effort to anticipate uncertain demands. Examples are railroads, which must distribute empty freight cars, and truckload motor carriers, which must supply empty tracks to different cities. Efficient allocation of vehicles, however, requires trying to anticipate future demands which must therefore be forecast, usually with considerable uncertainty. Even for most simple logistic dynamical system consisting of a wholesale store of capacity $\boldsymbol{Z}$, a retail store of capacity $\boldsymbol{Y}$ and automobiles which are taking part in goods delivery from a wholesale store to a retail store the author of paper [1] by means of imitation modeling succeeded in finding such a complex mode of the operation as limit cycles and other irregular attractors. But in reality any transport logistics model is dependent at random demand and operates at random environment. Besides, a time moment for restocking of goods also is a random value. This means that for quantitative analysis for goods growth we have to calculate not only given by deterministic dynamical system stock level of goods bet also to estimate possible random deviations on these idealized representations. To do this in our paper we consider some complicate proposed in [1] deterministic model assuming that the demand be random and coming at random time moments.

Previous research in the area of stochastic transshipment problems is relatively sparse. In [5] author considered stochastic transportation problem where the flows from supply to demand are deterministic and with linear transportation costs. Demands are assumed to be stochastic with stockout and holding costs provided as inputs to the model. The objective is to minimize transportation costs and expected stockout and holding costs, producing a simple convex, nonlinear objective function is easily solved. The simplicity of the model arises from the fact that flows must be sent before the demands are known and only one-time period is considered.

Let us describe the model of paper [1] and our proposed stochastic model more detail. The expressed in paper [1] mathematical model for dynamical analysis of the above transport logistics scheme is system of three dimensional ordinary differential equations:

$$\begin{cases} \frac{d}{dt}x(t) = kz(t)(\mathrm{R} - y(t)), \\ \frac{d}{dt}y(t) = a\,\mathrm{R}^{-1}\,x(t)(\mathrm{R} - y(t)) - by(t), \\ \frac{d}{dt}z(t) = -h\,\mathrm{R}^{-1}\,x(t)(\mathrm{R} - y(t)) + c\mathrm{A}^{-1}(\mathrm{A} - z(t)). \end{cases} \tag{1}$$

with right part, that dependent on the number of involving in goods delivery transport $x(t)$ and stock levels of goods $y(t)$ and $z(t)$ in the corresponding stores. This model constructed under assumption that for any $t \geq 0$:

- the increments $\Delta x(t) := x(t+\Delta) - x(t)$ of involving in goods delivery number of trucks are proportional to Δ multiplied by to stock $z(t)$ at wholesale store and a number of vacancies $\mathrm{R} - y(t)$ at retail store;
- the increments $\Delta y(t) := y(t+\Delta) - y(t)$ of stock levels of goods are proportional to Δ multiplied by involving in goods delivery number of trucks $x(t)$, a number of vacancies $\mathrm{R} - y(t)$ at retail store, after deduction of ordering for goods $by(t)\Delta$;
- the increments $\Delta z(t) := z(t+\Delta) - z(t)$ of stock levels of goods are proportional to Δ multiplied by $x(t)$, a number of vacancies $\mathrm{A} - z(t)$ at wholesale store, after deduction of goods transportable from wholesale store to retail store.

To take into account random properties of demand for goods we have to model a demand at the time interval $[t, t+\Delta)$ as a random variable that can arrive or not with dependent on interval length probability. That is why we propose for dynamical analysis of the above logistic transportation scheme a stochastic model given by following finite-difference approximation:

$$\begin{aligned} x_\varepsilon(t_{k+1}) &= x_\varepsilon(t_k) + kz_\varepsilon(t_k)(\mathrm{R} - y_\varepsilon(t_k))\Delta, \\ y_\varepsilon(t_{k+1}) &= y_\varepsilon(t_k) + a\,\mathrm{R}^{-1}\,x_\varepsilon(t_k)(\mathrm{R} - y_\varepsilon(t_k))\Delta - \Delta_y(t_k), \\ z_\varepsilon(t_{k+1}) &= z_\varepsilon(t_k) - h\,\mathrm{R}^{-1}\,x_\varepsilon(t_k)(\mathrm{R} - y_\varepsilon(t_k))\Delta + c\mathrm{A}^{-1}(\mathrm{A} - z_\varepsilon(t_k)), \end{aligned} \tag{2}$$

where $t_k = k\Delta, k \in N$, ε is a small positive parameter, and $\Delta_y(t_k)$ is a random sequence defined by dependent on two identically independent distributed (i.i.d.) independent uniform R(0,1) distributed series $\{\xi_k, k \in N\}$ and exponentially distributed with parameter $\varepsilon^{-1}\Delta$ series $\{\eta_k, k \in N\}$.

This means that there are random time moments $\{\tau_k, k \in N\}$ when the trajectory for stock levels of goods $y_\varepsilon(t)$ has small jumps $\varepsilon b(\xi_k) y_\varepsilon(t_k)$ bet these jumps occur very close: $\forall k \in \mathrm{N} : \mathrm{E}\{\tau_{k+1} - \tau_k\} = \varepsilon$. The sample trajectories for equations (2) for $\varepsilon = 0.01, \Delta = 0.001, k = 1, h = 1, R = 10, \mathrm{c} = 1, A = 100, b(u) = 2bu$ and some values of initial conditions and parameters b are shown below.

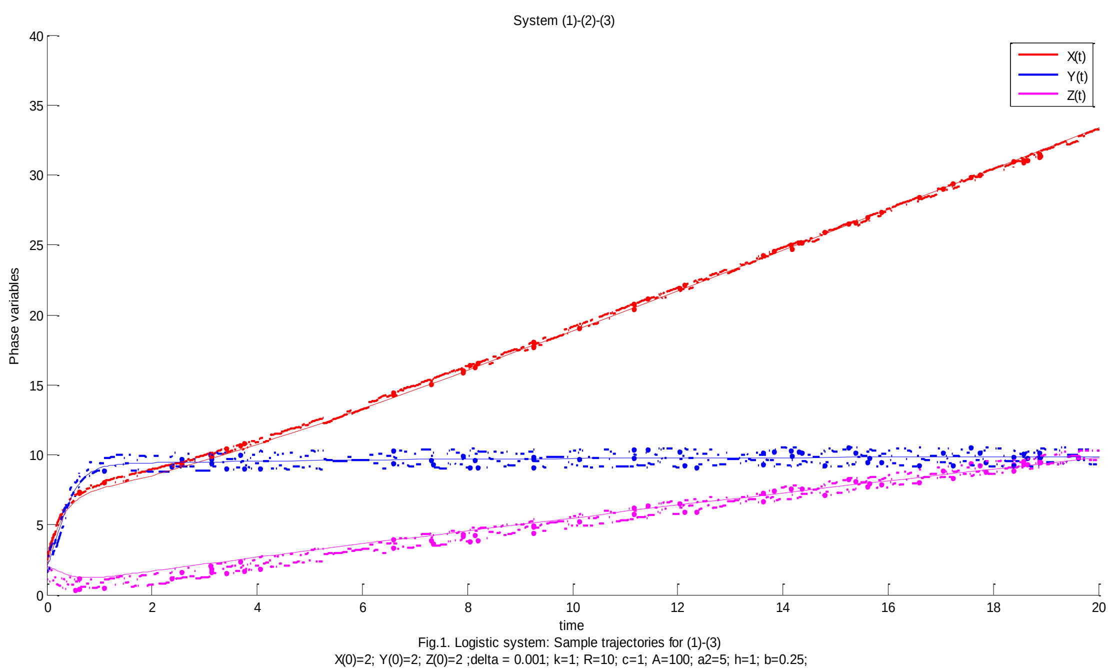

Fig.1.Sample trajectory for (1) - (2). $x_\varepsilon(0) = 2,\ y_\varepsilon(0) = 2,\ z_\varepsilon(0) = 2,\ b = 0.25.$

As we can see most dependent on random demand are dynamics for stock levels of goods $y_\varepsilon(t)$. At the next sections applying the stochastic averaging method [3] we derive approximatively solution for (2) as a three dimensional Gaussian process and discuss a behavior of mean value and variance for stock levels of goods $y_\varepsilon(t)$.

2 Diffusion approximation procedure

The defined in previous section stochastic dynamical system in more general form has been analyzed in our previous paper [2]. The corresponding to finite-difference equation (2) random process possess Markov property and may be analyzed through intermediary of generator [3]

$$\mathcal{L}(\varepsilon)v(x,y,z) := \lim_{\Delta\to 0}\frac{1}{\Delta}\mathrm{E}\{v(x_\varepsilon(t_k+\Delta), y_\varepsilon(t_k+\Delta), z_\varepsilon(t_k+\Delta)) - v(x,y,z)\big|_{x_\varepsilon(t_k)=x, y_\varepsilon(t_k)=y, z_\varepsilon(t_k)=z}\} =$$

$$= \left\{kz(\mathrm{R}-y)\frac{\partial}{\partial x} + a\,\mathrm{R}^{-1}x(\mathrm{R}-y)\frac{\partial}{\partial y} + [c\mathrm{A}^{-1}(\mathrm{A}-z) - h\,\mathrm{R}^{-1}x(\mathrm{R}-y)]\frac{\partial}{\partial z}\right\}v(x,y,z) + \tag{3}$$

$$+\frac{1}{\varepsilon}\int_0^1 [v(x, y+\varepsilon b(u)y, z) - v(x,y,z)]du$$

where $v(x,y,z)$ is an arbitrary sufficiently smooth bounded function . Now we have to derive a limit $\lim_{\varepsilon\to 0}\mathcal{L}(\varepsilon)v(x,y,z) := \mathcal{L}v(x,y,z)$, where

$$\mathcal{L} = kz(\mathrm{R}-y)\frac{\partial}{\partial x} + \left[a\,\mathrm{R}^{-1}x(\mathrm{R}-y) - by\right]\frac{\partial}{\partial y} + [c\mathrm{A}^{-1}(\mathrm{A}-z) - h\,\mathrm{R}^{-1}x(\mathrm{R}-y)]\frac{\partial}{\partial z} \tag{4}$$

and $b = \int_0^1 b(u)du$. The operator (4) correspond to dynamical system (1) and therefore for sufficiently small $\varepsilon > 0$ a behavior of defined by finite-difference equation (2) random dynamical system we can approximate by solution of equation (1), that is, if $x_\varepsilon(0) = x(0), y_\varepsilon(0) = y(0), z_\varepsilon(0) = z(0),$ then for any $T > 0$

$$\mathrm{P}\left\{\lim_{\varepsilon\to 0}\sup_{0\le t\le T}[|x_\varepsilon(t) - x(t)| + |y_\varepsilon(t) - y(t)| + |z_\varepsilon(t) - z(t)|] = 0\right\} = 1 \tag{5}$$

As it has been prove in [2] the deviations of solutions (2) on corresponding solutions of (1) have an order $\sqrt{\varepsilon}$ and we may analyze these deviations applying diffusion approximation procedure to no homogeneous three dimensional Markov process

$$X_\varepsilon(t) = \frac{x_\varepsilon(t) - x(t)}{\sqrt{\varepsilon}}, Y_\varepsilon(t) = \frac{y_\varepsilon(t) - y(t)}{\sqrt{\varepsilon}}, Z_\varepsilon(t) = \frac{z_\varepsilon(t) - z(t)}{\sqrt{\varepsilon}} \tag{6}$$

with zero initial conditions. The same as before we should derive a generator for (6)

$$\Lambda(\varepsilon)v(X,Y,Z) := \lim_{\Delta\to 0}\frac{1}{\Delta}\mathrm{E}\{v(X_\varepsilon(t_k+\Delta), Y_\varepsilon(t_k+\Delta), Z_\varepsilon(t_k+\Delta)) - v(X,Y,Z)\big|_{X_\varepsilon(t_k)=X, Y_\varepsilon(t_k)=Y, Z_\varepsilon(t_k)=Z}\}$$

and pass to limit $\lim_{\varepsilon\to 0}\Lambda(\varepsilon)v(X,Y,Z) := \Lambda v(\mathrm{X},Y,Z)$, where

$$
\begin{aligned}
\Lambda v(\mathrm{X},Y,Z) = \Big\{ & [-kz(t)Y + (\mathrm{R}-y(t))Z]\frac{\partial}{\partial X} + \\
& +[-a\,\mathrm{R}^{-1}\,x(t)Y + a\,\mathrm{R}^{-1}\,X(\mathrm{R}-y(t))]\frac{\partial}{\partial Y} + \frac{1}{2}\beta^2 y^2(t)\frac{d^2}{dY^2} + \\
& +[-c\mathrm{A}^{-1}Z + h\,\mathrm{R}^{-1}\,x(t)Y - X(\mathrm{R}-y(t))]\frac{\partial}{\partial Z}\Big\} v(\mathrm{X},Y,Z)
\end{aligned} \tag{7}
$$

and

$$
\beta^2 = \int_0^1 b^2(u)du\,. \tag{8}
$$

This operator identifies no homogeneous Markov process $\{X(t),Y(t),Z(t),t\geq 0\}$ which satisfies to systems of two ordinary equations and one stochastic Ito equation [3]:

$$
\frac{d}{dt}X(t) = -kz(t)Y(t) + (\mathrm{R}-y(t))Z(t), \tag{9}
$$

$$
dY(t) = [-a\,\mathrm{R}^{-1}\,x(t)Y(t) + a\,\mathrm{R}^{-1}\,X(t)(\mathrm{R}-y(t))]dt + \beta y(t)dw(t), \tag{10}
$$

$$
\frac{d}{dt}Z(t) = -c\mathrm{A}^{-1}Z(t) + h\,\mathrm{R}^{-1}\,x(t)Y(t) - X(t)(\mathrm{R}-y(t)), \tag{11}
$$

with initial conditions $\{X(0)=0, Y(0)=0, Z(0)=0\}$. As it has been proved in [2] finite dimensional distributions of the defined by equations (2) Markov process $\{x_\varepsilon(t), y_\varepsilon(t), z_\varepsilon(t)\}$ may be approximated by corresponding finite dimensional distributions of the process

$$
\bar{x}_\varepsilon(t) = x(t) + \sqrt{\varepsilon}X(t),\ \bar{y}_\varepsilon(t) = y(t) + \sqrt{\varepsilon}Y(t),\ \bar{z}_\varepsilon(t) = z(t) + \sqrt{\varepsilon}Z(t). \tag{12}
$$

Unfortunately, we cannot analyze variance separately approximation for stock levels of goods given by equation (10). We have to derive and solve the system of differential equations for all elements of a covariance matrix for the three dimensional Gaussian random vector $\{X(t),Y(t),Z(t)\}$:

$$
\begin{aligned}
&\frac{d}{dt}q_{XX}(t) = -2kz(t)q_{XY}(t) + (\mathrm{R}-y(t))q_{XZ}(t),\\
&\frac{d}{dt}q_{XY}(t) = -kz(t)q_{YY}(t) + (\mathrm{R}-y(t))q_{YZ}(t) - a\,\mathrm{R}^{-1}\,x(t)q_{XY}(t) + a\,\mathrm{R}^{-1}\,q_{XX}(t)(\mathrm{R}-y(t)),\\
&\frac{d}{dt}q_{XZ}(t) = -kz(t)q_{YZ}(t) + (\mathrm{R}-y(t))q_{ZZ}(t) - c\mathrm{A}^{-1}q_{XZ}(t) + h\,\mathrm{R}^{-1}\,x(t)q_{XY}(t) - q_{XX}(t)(\mathrm{R}-y(t)),\\
&\frac{d}{dt}q_{YY}(t) = -2a\,\mathrm{R}^{-1}\,x(t)q_{YY}(t) + 2a\,\mathrm{R}^{-1}\,q_{XY}(t)(\mathrm{R}-y(t)) + \beta^2 y^2(t),\\
&\frac{d}{dt}q_{YZ}(t) = -(a\,\mathrm{R}^{-1} + c\mathrm{A}^{-1})x(t)q_{YZ}(t) + a\,\mathrm{R}^{-1}\,q_{XZ}(t)(\mathrm{R}-y(t)) + h\,\mathrm{R}^{-1}\,x(t)q_{YY}(t) - q_{XY}(t)(\mathrm{R}-y(t)),\\
&\frac{d}{dt}q_{ZZ}(t) = -2c\mathrm{A}^{-1}q_{ZZ}(t) + 2h\,\mathrm{R}^{-1}\,x(t)q_{YZ}(t) - 2q_{XZ}(t)(\mathrm{R}-y(t)),
\end{aligned} \tag{13}
$$

with zero initial conditions. Applying the Runge-Kutta method for solution of equations (13) we calculate approximation for variance $\varepsilon q_{XY}(t)$ of stock levels of goods for $\varepsilon = 0.01, h = 1, R = 10, \mathrm{c} = 1, a = 5$ $A = 100, b(u) = 2bu$ with the same as in Fig.1 values of initial conditions and parameters b.

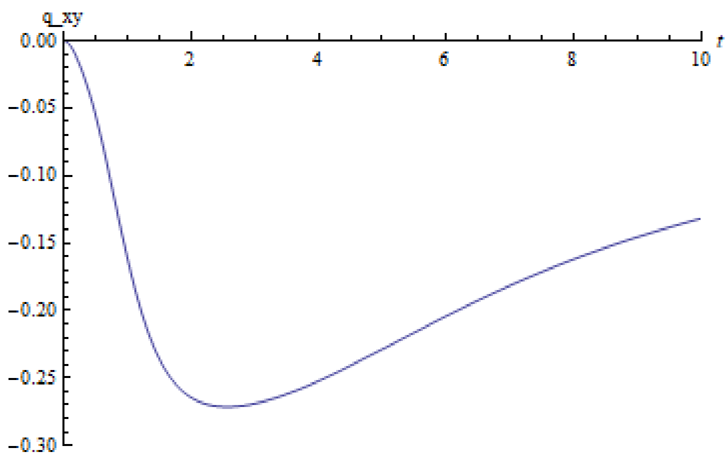

Fig.2. Variance $\varepsilon q_{XY}(t)$, for $x_\varepsilon(0) = 2,\ y_\varepsilon(0) = 2,\ z_\varepsilon(0) = 2,\ b = 0.25.$

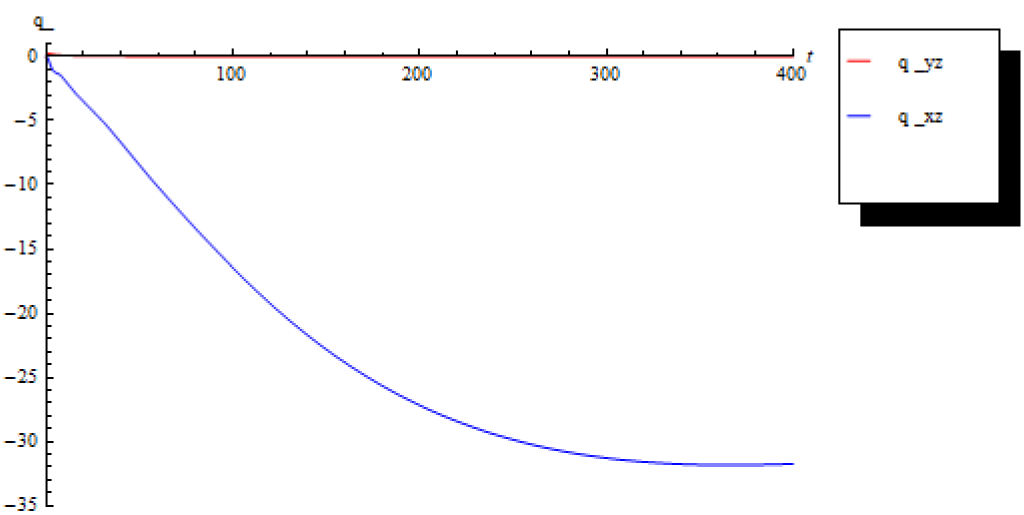

Fig.3. Other variances $\varepsilon q_{XZ}(t), \varepsilon q_{YZ}(t)$, for $x_\varepsilon(0) = 2,\ y_\varepsilon(0) = 2,\ z_\varepsilon(0) = 2,\ b = 0.25.$

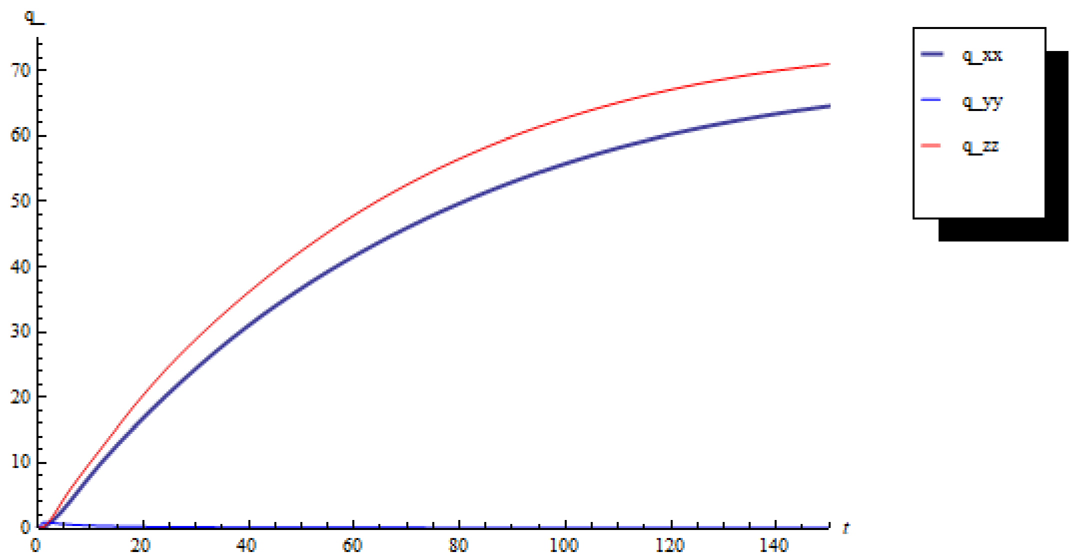

Fig.4. Variances $\varepsilon q_{XX}(t), \varepsilon q_{YY}(t), \varepsilon q_{ZZ}(t)$ for $x_\varepsilon(0) = 2,\ y_\varepsilon(0) = 2,\ z_\varepsilon(0) = 2, b = 0.25.$

As we can see all above variances converges to constant values, hence the system (1) - (2) - (3) will be stable.

3 Conclusion

The considered model is of the class related to the dissipative dynamic systems expressed in the form of nonlinear ordinary differential equations with stochastic coefficients. They allow us to consider the following important features of transport systems:

1) a combination of deterministic and stochastic factors of functioning;

2) the collective nature of the transport systems' functionality (the large amount of vehicles; numerous processes in transport systems);

3) the non-equilibrium state of the open logistic system in the form of persisting positive inventory.

References

[1] AGUREEV, I.E., ATLAS, E.E. Chaotic dynamics in transport systems, *Compexity.Mind.Postnonclassic*, eLIBRARY.RU, 2012, Nr.1, 94 - 106, (rus.), internet link: http://cmp.esrae.ru/pdf/2012/1/32.pdf .

[2] TSARKOV, Y.F. (CARKOVS, J.) Asymptotic methods for stability analysis Markov impulse dynamical systems, *Nonlinear dynamics and system theory*, 9 (2002), no. 2(1), 103–115.

[3] GIKHMAN, I.I. , SKOROKHOD, A.V. *Stochastic Differential Equations*, Springer-Verlag, 1972.

[4] SKOROKHOD, A.V. *Asymptotic methods in the theory of stochastic differential equations* (Transl. Math. Monographs **78**), American Mathematical Society,

Providence, RI, 1989.
[5] BLANK, M. Dynamics of traffic flow traffic jams: order and chaos // *Mosc. Math. J.*, 2000, 1:1, p.1–26.
[6] GODFREY, G. & POWELL, W. B. (2002a), An adaptive, dynamic programming algorithm for stochastic resource allocation problems I: Single period travel times, *Transportation Science* 36(1), 21–39.
[7] GODFREY, G. & POWELL, W. B. (2002b), An adaptive, dynamic programming algorithm for stochastic resource allocation problems II: Multi-period travel times, *Transportation Science* 36(1), 40–54.
[8] POWELL, W. B. (1986), A stochastic model of the dynamic vehicle allocation problem, *Transportation Science 20, 117–129.*

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