Introduction Into the Haar Like Transforms Based on Rotation Angles

Peteris Misans

***Abstract*—A novel class of generalization of Haar like discrete orthonormal functions is presented. The basics of the class of RABOT-Haar transforms (RA-HT) are described. Additionally, three novel subclasses of RA-HT are introduced using restrictions on the values of angles. The subclasses differ by the diversity of values of rotation angles. Examples of shapes of basis function(s) (BF) are presented. Basic properties of BFs for two subclasses of transforms are described. Certain parts of given BFs are invariant to shifting. Recurrent formulas for the calculation of BFs are given. An insertion of permutation matrices between factorized matrices ensures the ordering of BFs by ranks. The rules for the building of permutation matrices are formulated. The number of operations for the fast calculation of transforms is presented.**

***Index Terms*— Orthonormal Transforms, Parameterization of Transforms, Haar Functions, Generalization of Haar Functions**

I. Introduction

In the reference list in [1] we can find out there are about 300 papers concerning the signal processing based on the Haar functions (or Haar transforms – HT) or HT extensions at least. The goal of this paper is not to present a comprehensive overview or the analysis of history of Haar functions. Such overview is kept for the future. This paper deals only with one of the possible generalizations of HT.

The applications of HT cover a wide range of use, from the 1D signal de-noising to the edge detection, and from the design of digital devices to image processing. Although the applications of HT have a long and successful history across the last three decades, there are also many very recent works. For example, [2] highlights the using of HT for the identification of sparse impulse response by adaptive algorithm in Haar domain. There are many papers on the using of HT or its extensions in adaptive filtering (see, for example, [3]). Pogossova et al. [4] apply tree-structured HT (TSHT) to de-noising of signal. This indicates that novel research approaches dealing with HT may be very welcome. In the development of the extensions of classical HT it seems that the first trial of generalization of Haar functions is Watari functions [5]. B. Falkowski introduced a multi-polarity HT [6] that is useful for design of digital devices. [7] speaks about the multi-valued Vilenkin-Chrenstenson and Galois-Field Transforms. Further development of multi-valued Haar-like transforms has been continued by Stankovic [8]. K. Egiazarian in [9] introduced TSHT. In the context of the current paper, the short paper presented by B. Fino and R. Algazi [10] could be important. The authors tell about the class of Slant Haar transform (SHT) that is derived from the elementary rotation matrices by using generalized Kronecker product and some sequence of rotation angles. The idea about the rotation of planes in the Euclidian space is well known. This approach has been used, for example, in the QR-algorithm [11]. Unfortunately, the „DSP peoples“ use rotation angles as an interpretative instrument not so often. An exception is so called „parameterization“, for example, regarding to CORDIC based wavelet filters [12]. In [1] there are references to a dozen of papers at least about parameterization regarding orthogonal transforms.

In [13] we introduced the class of the **R**otation **A**ngle **B**ased **O**rthogonal **T**ransform (RABOT)-Haar-like transforms (RA-HT). However, only a very brief description of those functions was provided, and, in fact, this class of transforms was called the Haar-RABOT transforms. The name changed in the scope of this paper is more acceptable because of the hierarchy reasons – Haar functions can be treated as a subclass of RABOT functions. From the one point of view, we can treat defined functions as "damaged" Haar functions, but from the other side – as some kind of generalization of HT. Also, a further extension of class RA-HT in the current paper contributes for the name change. Generalization presented in [13] and below differs from the generalizations of HT defined in [10] and [8]. It is supposed that the transforms presented below are unknown before. The paper does not aim to introduce mathematically perfect definitions of functions. This task is for the future. Here RA-HT is described from the practical point of view only, without invoking serious mathematical definitions and investigations.

II. Basics of RA-Haar transforms

Let’s repeat some formulas from [13] for better understanding of the content of the paper presented below. The basic expression for the definition of matrix of RABOT transforms is given in [13]-(9). Below in formula (1) is provided a simplified expression of [13]-(9). For a wide class of orthonormal transforms the transform matrix can be

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P. Misans is with the Faculty of Electronics and Telecommunications, Riga Technical University, Riga, Latvia (e-mail: misans@ rsf.rtu.lv).

represented by the product of sparse orthonormal matrices:

$$\boldsymbol{\Phi}(\boldsymbol{\varphi}) = \mathbf{B}(\boldsymbol{\varphi}_l)\cdot \ldots \ \cdot \mathbf{B}(\boldsymbol{\varphi}_j)\cdot \ldots \cdot \mathbf{B}(\boldsymbol{\varphi}_1) \qquad (1)$$

where $\boldsymbol{\varphi}_\mathbf{j}$ – j-th column of the angle matrix:

$$\boldsymbol{\varphi} = \begin{bmatrix} \phi_{1\,1} & \phi_{1\,2} & \phi_{1\,3} & \cdots & \phi_{1\,l} \\ \phi_{2\,1} & \phi_{2\,2} & \phi_{2\,3} & \cdots & \phi_{2\,l} \\ \phi_{3\,1} & \phi_{3\,2} & \phi_{3\,3} & \cdots & \phi_{3\,l} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_{\frac{N}{2},\,1} & \phi_{\frac{N}{2},\,2} & \phi_{\frac{N}{2},\,3} & \cdots & \phi_{\frac{N}{2},\,l} \end{bmatrix}, \qquad (2)$$

$N=2^l$ – the size of transform matrix. Formula (2) is the same as [13]-(7) with a slight change of indexes. The matrix **B** is a sparse **S**tairs-like **O**rthonormal **G**eneralized **R**otation **M**atrix (SOGRM – [13]). The sparseness of **B** in (1) ensures the existence of fast algorithm for the calculation of RABOT transform [13].

A subclass of RA-HT is defined in [13]. Some limitations on rotation angles ([13]–(10)) and the structure of angle matrix (2) are used. In [13] we operate only with two values of angles and the stair-like structure of angle matrix. In the present paper we limit the value of angle ϕ_2 to zero but allow arbitrary and different values of ϕ_1 for each SOGRM matrix in (1):

$$\boldsymbol{\varphi}_j = \begin{cases} \phi_{i\,j}, & j=1, \qquad\qquad i\in[1, N/2] \\ \phi_{i\,j}, & j>1, \qquad i\in[1, N/2 - N/(2^j)] \\ 0, & j>1, \quad i\in[N/2 - N/(2^j)+1, N/2] \end{cases} \qquad (3)$$

where $\phi\in[0, 2\pi]$ in general case. The choice of zero angles is explained by some similarity of shapes of RA-HT BFs to the shapes of classical Haar functions in zero-valued segments. The extension when instead of zero value angle a nonzero value angle is used we keep for further investigations. The mutual exchange of places of angles between the second and the third row in (3) leads to reordering of rows (i.e. BFs) in transform matrix $\boldsymbol{\Phi}$, comparing (3) in this paper and (10) in [13]. A significant property of angle matrix (2) for RA-HT is a decreased number of nonzero angles across the columns of this matrix. For each next column, when moving from left to right, this number decreases twice.

Example 1: For N=8 we have:

$$\boldsymbol{\varphi} = \begin{bmatrix} \boldsymbol{\varphi}_1 & \boldsymbol{\varphi}_2 & \boldsymbol{\varphi}_3 \end{bmatrix} = \begin{bmatrix} \phi_{1\,1} & \phi_{1\,2} & \phi_{1\,3} \\ \phi_{2\,1} & \phi_{2\,2} & 0 \\ \phi_{3\,1} & 0 & 0 \\ \phi_{4\,1} & 0 & 0 \end{bmatrix} \qquad (4)$$

In the case when all nonzero angles $\phi_{i\,j}=\pi/4$, it is the classical HT.

RA-HT (and classical orthonormal HT also) can be presented by the product of three sparse matrices:

$$\boldsymbol{\Phi}_8(\boldsymbol{\varphi}) = \mathbf{B}_8(\boldsymbol{\varphi}_3)\cdot \mathbf{B}_8(\boldsymbol{\varphi}_2)\cdot \mathbf{B}_8(\boldsymbol{\varphi}_1) \qquad (5)$$

where:

$$\mathbf{B}_8(\boldsymbol{\varphi}_3) = \begin{bmatrix} \mathbf{s}_{13} & \mathbf{c}_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \mathbf{c}_{13} & -\mathbf{s}_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{B}_8(\boldsymbol{\varphi}_2) = \begin{bmatrix} s_{12} & c_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_{22} & c_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ c_{12} & -s_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{22} & -s_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad (6)$$

$$\mathbf{B}_8(\boldsymbol{\varphi}_1) = \begin{bmatrix} s_{11} & c_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_{21} & c_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{31} & c_{31} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_{41} & c_{41} \\ c_{11} & -s_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{21} & -s_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{31} & -s_{31} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{41} & -s_{41} \end{bmatrix}$$

Here s_{ij}, c_{ij} – shortcuts for $sin(\phi_{ij})$ and $cos(\phi_{ij})$, respectively. $\mathbf{B_8(\boldsymbol{\varphi}_1)}$ (SOGRM) matrix is built like the elementary rotation structures in [13] called as $\mathbf{R}_+$ elementary rotation matrices.

For the leftward terms in (5) more simplified structures of SOGRMs may be used. The simplification is caused by using zero values for certain angles. The most leftward term $\mathbf{B}_8(\boldsymbol{\varphi}_3)$ contains only one rotation structure (bolded in formula).

Example 2: For **N=4** we obtain (7). It shows that the structure of RA-HT matrix is the same as for the classical HT, ignoring the ordering of matrix rows. The transform matrix (7) represents the set of orthonormal BFs. By varying of angles an infinite number of transforms with HT-like structures can be obtained (in the sense of zero value segments).

$$\begin{aligned} \boldsymbol{\Phi}_4(\boldsymbol{\varphi}) &= \mathbf{B}_4(\boldsymbol{\varphi}_2)\cdot \mathbf{B}_1(\boldsymbol{\varphi}_1) = \\ &= \begin{bmatrix} s_{12} & c_{12} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ c_{12} & -s_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} s_{11} & c_{11} & 0 & 0 \\ 0 & 0 & s_{21} & c_{21} \\ c_{11} & -s_{11} & 0 & 0 \\ 0 & 0 & c_{21} & -s_{21} \end{bmatrix} = \\ &= \begin{bmatrix} s_{12}\cdot s_{11} & s_{12}\cdot c_{11} & c_{12}\cdot s_{21} & c_{12}\cdot c_{21} \\ 0 & 0 & c_{21} & -s_{21} \\ c_{12}\cdot s_{11} & c_{12}\cdot c_{11} & -s_{12}\cdot s_{21} & -s_{12}\cdot c_{21} \\ c_{11} & -s_{11} & 0 & 0 \end{bmatrix} \end{aligned} \qquad (7)$$

III. ORDERING OF BASIS FUNCTIONS

Describing angle-based transforms by the product of

SOGRM matrices simplifies many things from the methodical point of view but it is not sufficient in some particular cases. For example, trying to compare conventional Haar functions with RA-HT functions we meet with some difficulties because of different ordering of BFs in the transform matrix.

We should reorder the rows of each factorized matrix **B** in (1) to get the by-rank-ordered RA-HT matrix:

$$\mathbf{\Phi}_r(\boldsymbol{\varphi}) = \mathbf{B}_r(\boldsymbol{\varphi}_l)\cdot \ldots \cdot \mathbf{B}_r(\boldsymbol{\varphi}_j)\cdot \ldots \cdot \mathbf{B}_r(\boldsymbol{\varphi}_1) \qquad (8)$$

where index r indicates "ordering by ranks". It is well known that a permutation of rows of matrix can be presented as a multiplication by the permutation matrix on the left:

$$\mathbf{B}_r(\boldsymbol{\varphi}_k) = \mathbf{P}_{rk}\cdot \mathbf{B}(\boldsymbol{\varphi}_k) \qquad (9)$$

The matrix $\mathbf{P}_{r\,k}$ must be ordered in a way to move all ‘1’s of **B** to the diagonal of factorized matrix $\mathbf{B}_r$. From the theory of fast HT it is well known that factorized matrices have the structure with a SOGRM like sub-matrix in the left-upper corner and a unit matrix like structure in the right-upper corner.

$$\mathbf{B}_{N\,r\,k} = \begin{bmatrix} * & * & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & * & * & 0 & 0 & \cdots & 0 \\ * & * & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & * & * & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (10)$$

The sizes of the mentioned substructures depend on the index of the used factorized matrix.

The main goal of the use of permutation matrix is the transformation of SOGRM structure to the structure shown in (10). In the next example we try to reorder $\mathbf{B_8(\boldsymbol{\varphi}_2)}$ (6) to the matrix like (10).

Example 3: For example, for **N**=8, k=2 the following permutation matrix is used:

$$\mathbf{P}_{8\,r\,2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (11)$$

After permutation of rows of $\mathbf{B}_2$ it results in the reordered matrix (12).

$$\mathbf{B}_{8\,r}(\boldsymbol{\varphi}_2) = \begin{bmatrix} s_{12} & c_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_{22} & c_{22} & 0 & 0 & 0 & 0 \\ c_{12} & -s_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{22} & -s_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (12)$$

A. Rules for the Building of Permutation Matrix

Proof of the rules for the building of permutation matrices can be missed because of clear evidence. These rules follow directly from the stair-like structure of SOGRM matrix and from the restrictions (19), (29), and (33) on the angle matrices presented below. Additionally, we have an experimental proof of rules listed below for **N** up to **8192**. The textual explanations given below are brief and have some intersections. A reader should use formulas (not the textual formulations) for accurate applications (for example, programming).

Let’s the size of permutation matrix to be N-by-N and the index $\mathbf{k \in [1, \log_2(N)]}$. Each permutation matrix can be built-up if we follow the rules listed below.

<u>*Rule 1.*</u>: The indexes of first $N/2^k$ nonzero elements are equal. For example, these elements are diagonal elements:

$$\mathbf{P}_{Nrk}(i,j) = \begin{cases} 1, \; i=j \\ 0, \; i \neq j \end{cases}, \quad for \; \begin{cases} i,j \in [1,N], & if\, k=1 \\ i,j \in [1, N/2^k], & if\, k>1 \end{cases} \qquad (13)$$

where i – row index, j – column index.

<u>*Rule 2.*</u>: Nonzero elements in the left-half of permutation matrix have the following row column indexes:

$$\mathbf{P}_{Nrk}(i,j) = \begin{cases} 1, \; i = 2\cdot j \\ 0, \; i \neq 2\cdot j \end{cases}, \quad j \in [\frac{N}{2^k}+1, \frac{N}{2}] \; for \; k \geq 2 \qquad (14)$$

<u>*Rule 3.*</u>: Nonzero elements in the upper-half of permutation matrix have the following row and column indexes:

$$\mathbf{P}_{Nrk}(i,j) = \begin{cases} 1, \; j = i + \frac{N}{2} - \frac{N}{2^k} \\ 0, \; j \neq i + \frac{N}{2} - \frac{N}{2^k} \end{cases}, \quad i \in [\frac{N}{2^k}+1, \frac{N}{2}] \; for \; k \geq 2 \qquad (15)$$

<u>*Rule 4.*</u>: Nonzero elements in the right-half of permutation matrix have the following row and column indexes:

$$\mathbf{P}_{Nrk}(i,j) = \begin{cases} 1, i = 2\cdot(j - \frac{N}{2}) - 1 \\ 0, i \neq 2\cdot(j-\frac{N}{2}) - 1 \end{cases}, \quad j \in [\frac{N}{2} + \frac{N}{2^k} + 1, N] \; for \; k \geq 2 \qquad (16)$$

A reader should also remember that the basic property of permutation matrix is – "one nonzero element per row or column only". It is important when programming (13)–(16).

Example 4: For **N**=4 (see Example 2, Formula (7)) only one permutation of the leftward term is needed. A permutation of rightward matrix is not performed at all because the permutation matrix $\mathbf{P}_1$ is a unit matrix. That follows from Rule 1. In such a way we have a rank-ordered RA-Haar transform matrix in the contrast to example (7).

$$\mathbf{\Phi}_{4r}(\boldsymbol{\varphi}) = \mathbf{P}_{4r2} \cdot \mathbf{B}_4(\boldsymbol{\varphi}_2) \cdot \mathbf{B}_1(\boldsymbol{\varphi}_1) =$$

$$= \begin{bmatrix} s_{12} & c_{12} & 0 & 0 \\ c_{12} & -s_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_{11} & c_{11} & 0 & 0 \\ 0 & 0 & s_{21} & c_{21} \\ c_{11} & -s_{11} & 0 & 0 \\ 0 & 0 & c_{21} & -s_{21} \end{bmatrix} = \quad (17)$$

$$= \begin{bmatrix} s_{12}\cdot s_{11} & s_{12}\cdot c_{11} & c_{12}\cdot s_{21} & c_{12}\cdot c_{21} \\ c_{12}\cdot s_{11} & c_{12}\cdot c_{11} & -s_{12}\cdot s_{21} & -s_{12}\cdot c_{21} \\ c_{11} & -s_{11} & 0 & 0 \\ 0 & 0 & c_{21} & -s_{21} \end{bmatrix}$$

IV. Novel subclasses of RA-Haar transforms

We see here a freedom for the variety of limitations on the diversity of rotation angles. Each specific restriction can lead to the definition of a novel class of transforms. Some of these limitations are analyzed below.

A. *Subclass of RA-HT with Constant Rotation Angle*

The transform has been introduced in [13] as Haar-RABOT transform but the description presented there is very brief.

A typical limitation on the values of nonzero angles follows from accepting equal angles in (3). For example, if **N=16**, the size of angle matrix is 8x4 with the following structure (note that we present the transposed version of matrix):

$$\boldsymbol{\varphi}^T = \begin{bmatrix} \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & 0 & 0 & 0 & 0 \\ \phi & \phi & 0 & 0 & 0 & 0 & 0 & 0 \\ \phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (18)$$

Or, in the general case, more correct condition for the angles used in context with (3) is:

$$\boldsymbol{\varphi}_j = \begin{cases} \phi, & j=1, \quad i\in[1, N/2] \\ \phi, & j>1, \quad i\in[1, N/2 - N/(2^j)] \\ 0, & j>1, \quad i\in[N/2 - N/(2^j)+1, N/2] \end{cases} \quad (19)$$

By varying of angle it is possible to obtain an infinite number of transforms. Further we call this class of transforms the RA-HT with **C**onstant **R**otation **A**ngle (CRA-HT). The term "constant" here means the "mutual" constancy of nonzero angles inside the angle matrix.

The CRA-HT transform should not be confused with CRAFOT transform (introduced in [14]) that operates with one angle only. The transform introduced now uses an additional zero angle (except for $\mathbf{B}_1$), and the structures of angle matrix for both mentioned transforms differ.

Example 5: For $\phi=\pi/4$ it gives the simplest member of CRA-HT class – the well-known orthonormal HT. Here a permutation matrix is applied for the reordering of BFs. In the case of classical HT corresponding scaling factors should be used in (20). Here we ignore these terms.

$$\mathbf{\Phi}_{4r}(\boldsymbol{\varphi})) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \mathbf{B}_4([\frac{\pi}{4}, 0]) \cdot$$

$$\cdot \mathbf{B}_4([\frac{\pi}{4}, \frac{\pi}{4}]) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \quad (20)$$

where

$$\mathbf{B}_4([\frac{\pi}{4}, \frac{\pi}{4}]) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix},$$

$$\mathbf{B}_4([\frac{\pi}{4}, 0]) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (21)$$

Each of Haar BFs can be presented by three or less values. Other members of CRA-HT (for $\phi\neq\pi/4$) have more complicated amplitude structure.

Example 6: Next figure presents first four CRA-HT BFs (ordered by ranks) in the stairs-like graphical interpretation.

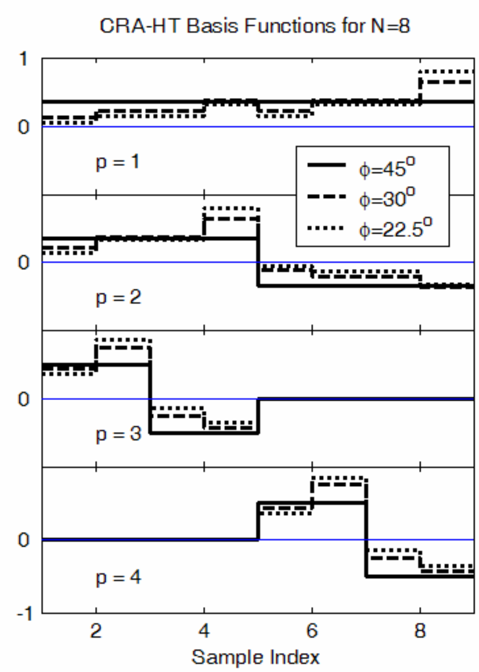

Fig. 1. Shapes of the first four discrete CRA-HT BFs for certain values of rotation angles and N=8

This shows that for the rotation angle ϕ=45° the CRA-HT class of transforms give the well-known set of orthonormal Haar BFs (solid line). By decreasing the angle value toward zero we observe the change of shapes to the well-known orthogonal unit pulses.

1) Basic Properties of CRA-HT

We observe here several differences in comparison with the classical Haar functions. First of all the first BF is not a pure DC component but has some oscillations. Other BFs contain both the DC and the oscillating component. Additionally, there is a more wide diversity of function values for each BF.

A common property of classical HT and CRA-HT is the existence of segments with zero-values.

Property 1. *CRA-HT matrix rows (i.e. BFs) with indexes*

$$p \in [2^k + 1,\ 2^{k+1}],\ where\ \ k \in [1, \log_2(N) - 1] \tag{22}$$

for a chosen ***k*** *have nonzero and zero values segments so that*

$$\mathbf{\Phi}_r(p, x) = \begin{cases} \neq 0,\ x \in [(m-1)\dfrac{N}{2^k} + 1,\ m\dfrac{N}{2^k}] \\ = 0,\ x \notin [(m-1)\dfrac{N}{2^k} + 1,\ m\dfrac{N}{2^k}] \end{cases} \tag{23}$$

$$where\ m = p - 2^k$$

For example, for BF $\mathbf{\Phi_{8\,r}}(6, x)$ there are nonzero values for $x\in[5, 6]$ (k=2, m=6–4=2).

Further we will use the term "primary basis function". We will call the first BF from the subset of BFs defined by the indexes (22) as the primary BF (PBF). This means that PBFs have indexes:

$$p_k = 2^k + 1,\ \ where\ \ k \in [0,\ \log_2(N) - 1] \\ (p_k = 2, 3, 5, 9, 17, 33, ...) \tag{24}$$

The nonzero parts of BFs shapes with indexes $p\in[3,4]$ in *Fig. 1* show the full equivalence. A similar mutual equality for the shapes of BFs with indexes $p\in[5,8]$ can be observed also. In other words, we can declare the property of "invariance-to-shifting".

Property 2. *CRA-HT matrix rows (i.e. BFs) with indexes* (22) *for a chosen* ***k*** *have equivalent nonzero parts of shapes of BFs. Any BF for* $p>2^k+1$ *can be derived by simple right shifting of PBF so that*:

$$\mathbf{H}(2^k + 1 + m,\ x) = \begin{cases} \mathbf{H}(2^k + 1,\ x - m\dfrac{N}{2^k}) \\ 0, x \in [m\dfrac{N}{2^k} + 1, N] \end{cases} \tag{25}$$

$$where\ m \in [1, 2^k - 1],\ \ k \in [1, \log_2(N) - 1]$$

The invariance of nonzero part of shape of BF to shifting is a well-known property of HT and wavelets.

The next important property of CRA-HT concerns the recurrent relations between PBFs ($p=2^k+1$).

Property 3. *CRA-HT rows with indexes* p_k *for* $k\in[1,\log_2(N)]$ *(i.e. PBFs) satisfy the following recurrent relation:*

$$\mathbf{H}(p_{k+1},\ x) = \begin{cases} c^{-1} \cdot \mathbf{H}(p_k,\ 2\cdot x),\ if\ \ x \in [1,\ N/2^{k+2}] \\ -c^{-1} \cdot \mathbf{H}(p_k,\ 2\cdot(x - \dfrac{N}{2^{k+2}}) - 1), \\ \quad if\ \ x \in [N/2^{k+2} + 1,\ N/2^{k+1}] \end{cases}, \tag{26}$$

if $c\neq0$. But, for $p\in[1, 2]$:

$$\mathbf{H}(2,\ x) = \begin{cases} \mathbf{H}(1,\ 2\cdot x),\ if\ \ x \in [1,\ N/2] \\ -\mathbf{H}(1,\ 2\cdot(x - \dfrac{N}{2}) - 1), \\ \quad if\ \ x \in [N/2 + 1,\ N] \end{cases} \tag{27}$$

B. Subclass of RA-HT with Constant Rotation Angle in Factorized Matrix

The next step in the extension of angle matrix (18) is the requirement for the constancy of angle within a row or a column of (2):

$$\boldsymbol{\varphi}^T = \begin{bmatrix} \phi_1 & \phi_1 & \phi_1 & \phi_1 & \phi_1 & \phi_1 & \phi_1 & \phi_1 \\ \phi_2 & \phi_2 & \phi_2 & \phi_2 & 0 & 0 & 0 & 0 \\ \phi_3 & \phi_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \phi_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{28}$$

A more accurate definition follows from (3):

$$\boldsymbol{\varphi}_j = \begin{cases} \phi_1, & j = 1, & i \in [1, N/2] \\ \phi_j, & j > 1, & i \in [1, N/2 - N/(2^j)] \\ 0, & j > 1, & i \in [N/2 - N/(2^j) + 1, N/2] \end{cases}. \tag{29}$$

Further we will call this transform the RA-HT with **C**onstant **R**otation **A**ngle **I**n **M**atrix (CRAIM-HT). There is some formal similarity to the CRAIMOT transform [13]. In the CRAIMOT transform we operate with the same angle within each of factorized matrices **B**. CRAIM-HT is based only on two angles (one of them has zero value) inside each of **B** matrices, except $\mathbf{B}_1$.

Example 7: In next figure we present first three CRAIM-HT BFs (ordered by ranks).

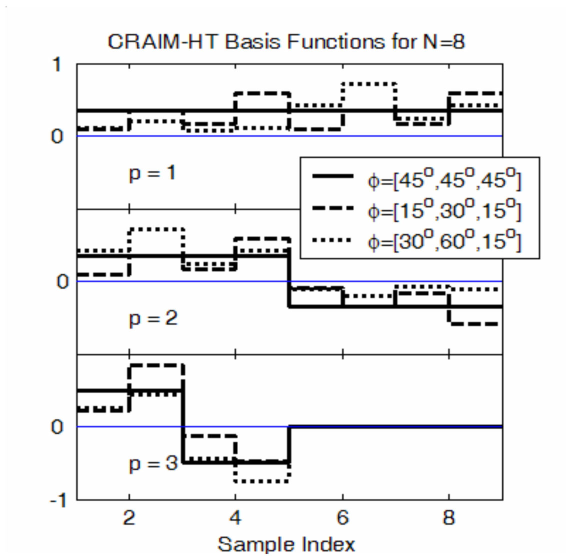

Fig. 2. Shapes of the first three discrete CRAIM-HT BFs for certain

values of rotation angles and N=8

From the results got by playing with the CRAIM-HT BFs generator, it is evident that although there are some common similarities with previous class of functions we have also some new features. The present functions are more "oscillative". Now we can get a more broad diversity of shapes in comparison with the CRA-HT class of functions.

1) Basic Properties of CRAIM-HT

For CRAIM-HT the first two properties (including expressions (22)-(25)) of CRA-HT apply also. The recurrent relations between PBFs ($p=2^k+1$) differ from the way it is for Property 3 of CRA-HT. That is because of extended diversity of angles used.

Next picture shows a simplified scheme to build up the current PBF from the previous PBF by the "cross-copying" of the halves of corresponding rows of transform matrix.

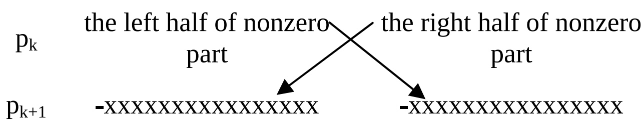

Fig. 3. Simplified rule for the building of PBF

The copying includes the change of sign and multiplication of taken parts of row by a constant (see below).

Property 4. *CRAIM-HT rows with indexes p_k for $k\in[0,\log_2(N)]$ (i.e. PBFs) satisfy the following recurrent relation:*

$$\mathbf{H}(p_{k+1},\ x) = \begin{cases} -b\cdot\mathbf{H}(p_k,\ x+N/2^{k+2}),\ if\ \ x\in[1,\ N/2^{k+2}], \\ \quad -b\cdot\mathbf{H}(p_k,\ x-\dfrac{N}{2^{k+2}})-1), \\ \quad if\ \ x\in[N/2^{k+2}+1,\ N/2^{k+1}] \end{cases} \tag{30}$$

if c≠0. But, for p∈[1, 2];

$$\mathbf{H}(2,\ x) = \begin{cases} \mathbf{H}(1,\ 2\cdot x),\ if\ \ x\in[1,\ N/2] \\ \quad -\mathbf{H}(1,\ 2\cdot(x-\dfrac{N}{2})-1), \\ \quad if\ \ x\in[N/2+1,\ N] \end{cases} \tag{31}$$

C. RA-HT with Reduced Sequences of Rotation Angles in Factorized Matrix

In the further generalization step we fill the columns of angle matrix by reduced (to half) angle sequences:

$$\boldsymbol{\varphi}^T = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \phi_7 & \phi_8 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 & 0 & 0 & 0 & 0 \\ \phi_1 & \phi_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \phi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{32}$$

The corresponding definition of the column of angle matrix is as follows:

$$\boldsymbol{\varphi}_j = \begin{cases} \phi_i, & j=1, \qquad\qquad i\in[1,N/2] \\ \phi_i, & j>1, \quad i\in[1,N/2-N/(2^j)] \\ 0, & j>1,\ \ i\in[N/2-N/(2^j)+1,N/2] \end{cases} \tag{33}$$

We will call this transform the RA-HT with **R**educed **S**equences of Rotation **A**ngles (RSA-HT). Here some formal and small similarity to the CRMOT [13] can be found in the definition of values of angles. But anyway – these transforms (RSA-HT and CRMOT) are very different.

Example 8: Next figure presents two of RSA-HT BFs.

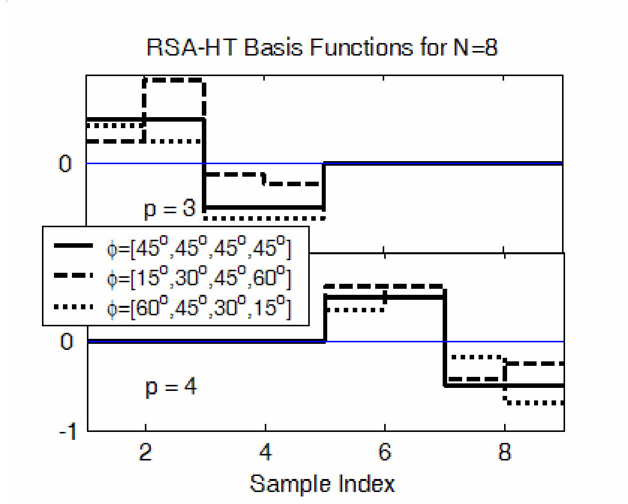

Fig. 4. Shapes of two discrete RSA-HT BFs (p=3,4) for certain values of rotation angles and N=8

Comparing the shapes of BFs to the corresponding BFs from the sets of functions defined in previous subsections, one principal difference appears – the nonzero parts of BFs with indexes p=2 and p=3 differ (non-invariant to shifting). We keep the investigations of RSA-HT properties for the future.

V. Number of Operations for the Calculation of RA-Haar Transforms

For the direct calculation of transform the RA-HT matrix can be used:

$$\mathbf{Y} = \mathbf{\Phi}_r\cdot\mathbf{X} \tag{34}$$

where **X** – input signal vector, **Y** – RA-Haar spectrum vector. As an alternative, for the fast calculation of spectrum a factorized form of (34) can be used:

$$\mathbf{Y} = \mathbf{B}_r(\boldsymbol{\varphi}_l)\cdot\ldots\ \cdot\mathbf{B}_r(\boldsymbol{\varphi}_j)\cdot\ldots\cdot\mathbf{B}_r(\boldsymbol{\varphi}_1)\cdot\mathbf{X} \tag{35}$$

Since the transform matrix $\mathbf{\Phi}_r$ contains a large number of zeros (>50% for N≥8), we need to compare the number of operations spent for the calculation of spectrum **Y** by (34) and the number got by (35).

A. Number of Operations for the Direct Calculation

For the transform matrix of size *N*-by-*N*, N^2 multiplications and $N\cdot(N-1)$ additions are needed to calculate the spectrum in a general case. Property 1 (declared above) allows a significant reduction of the number of operations. In such a way, if we take into account zero values, we obtain:

$$
\begin{aligned}
n_{mult\,\Phi} &= N^2 - \sum_{k=1}^{n-1} 2^k \cdot (N - N/2^k) = N \cdot (n+1) \\
n_{sum\,\Phi} &= N^2 - N - \sum_{k=1}^{n-1} 2^k \cdot (N - N/2^k) = N \cdot n
\end{aligned}
\tag{36}
$$

n_{mult} – the number of multiplications, n_{sum} – the number of additions. The total number of operations is:

$$n_{op\,\Phi} = n_{mult\,\Phi} + n_{sum\,\Phi} = N \cdot (2 \cdot n + 1)\,. \tag{37}$$

B. Number of Operations for the Fast Calculation

As mentioned above, the sparseness of $\mathbf{B}_r$ in (35) ensures a fast algorithm for the calculation of RA-HT. The total number of mathematical operations for the calculation of fast transforms by (35) may be easily found. Matrices $\mathbf{B}_r$ contain mainly zeros. There are usually only two elements per column and per row which are not equal to zero (see, for example, (6)). The RA-HT is a special case because of the annihilation of part of rotation structures within **B**r matrix (see, for example, (10)). The cause of annihilation is zero value angles. A single elementary rotation takes four multiplications and two summations in general case. If we go leftward through the factorized matrices, the number of operations reduces twice per each matrix. This means that the total number of operations can be expressed as:

$$n_{op\,FT} = n_{mult\,FT} + n_{sum\,FT} = 6 \cdot (N-1) \tag{38}$$

where

$$
\begin{aligned}
n_{mult\ FT} &= 4 \cdot \left(\frac{N}{2^1} + \frac{N}{2^2} + \ldots + \frac{N}{2^n}\right) = 4 \cdot (N-1) \\
n_{sum\ FT} &= 2 \cdot \left(\frac{N}{2^1} + \frac{N}{2^2} + \ldots + \frac{N}{2^n}\right) = 2 \cdot (N-1)
\end{aligned}
\tag{39}
$$

$n_{mult\ FT}$ – the number of multiplications for FT, $n_{sum\ FT}$ – the number of additions for FT.

Further reduction of the number of operations for CRAIM-HT and RSA-HT is obtained using some angles with values $\pi/4$ and corresponding scaling factors. It is also possible to avoid any multiplications, for example, in the case of rationalized HT (BFs have values +1, –1, and 0 only).

1) Efficiency Rate

We will use efficiency rate to compare the number of operations needed for the calculation of RA-HT by (34) and the number got by (35). For the practical estimation of rate asymptotic formulas may be used:

$$r_{ops\,RAHT} = \frac{n_{mult\ \Phi} + n_{sum\ \Phi}}{n_{mult\ FT} + n_{sum\ FT}} \approx \frac{n}{3} = \frac{\log_2(N)}{3} \tag{40}$$

In the case the rationalized HT is used the number of multiplications is zero and the ratio is higher:

$$r_{ops\,HT} = \frac{n_{sum\ \Phi}}{n_{sum\ FT}} = \frac{N \cdot n}{2 \cdot (N-1)} \approx \frac{n}{2} = \frac{\log_2(N)}{2} \tag{41}$$

Next figure shows the efficiency rate in dependence on the logarithm of N. The present curves reflect the accurate versions of (40) and (41). Practically linear dependencies appear for the large values of N as follows from the asymptotic formulas (most rightward formulas in (40) and (41)).

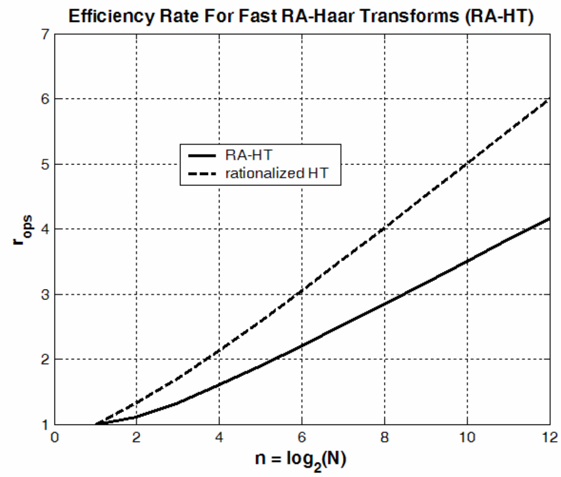

Fig. 5. Efficiency rate for the fast RA-HT and rationalized HT

The curves may be treated as the upper and lower limits for the efficiency rate.

VI. Conclusion

- The paper introduces a novel class of transforms with the Haar like structure of transform matrix. A formal similarity between the transforms introduced and the classical Haar transform appears as the equality to zero for the same segments of basis functions. The definition of presented transforms is based on the rotation angle approach described in [13].
- Three subclasses of Haar like transforms introduced here differ by the number of nonzero rotation angles used, and they have the same structure of angle matrix with zero value angles in certain parts of the matrix. The simplest is the CRA-HT which uses one nonzero angle only. The CRAIM-HT operates with $\log_2(N)$ nonzero angles. The Slant Haar transform may be treated as the special case of CRAIM-HT for some relations between angles. For the RSA-HT N/2 angles are used.
- For the CRA-HT and the CRAIM-HT so called primary basis functions are defined. The recurrent relations occur in primary basis functions. Nonzero segments for some basis functions can be formed by easily shifting nonzero segments of primary basis functions rightward.
- The rules for permutation of rows of factorized matrices are defined. The permutation of rows of SOGRM matrices ensures the ordering of basis functions by ranks.
- All defined transforms have fast algorithms. Efficiency rate is practically a linear function dependent on $\log_2(N)$

for large N. For N=4096 the fast algorithm needs four times less number of operations than the direct calculation of transform.

A. At This Moment And for the Future

This paper must be treated as introductory. We see a lot of work related to the present subject still. The first two years of experience with angle based transforms are very promising. While this paper was waiting for publication we presented the initial versions of RA-HT filters [16]. There we demonstrated that RA-HT could be very promising, for example, in edge detection. Our recent activities in the development of CRAIMOT based devices are reflected in [17]-[19]. We are working on the applications of RA-HT transforms and we have started with FPGA implementation of RA-HT functions. These functions could be concurrent to wavelets, for example, in image processing. It seems to be more beneficial as Haar functions. There is a lot of space for the special investigation and comparison of CRAIM-HT and Slant HT [10], and, for the comparison to wavelets, of course.

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