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A COMPARATIVE ANALYSIS OF ALTERNATIVE RULES OF BELIEF COMBINATION

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1. Introduction

Soon after the theory of evidence appeared in 1976, its main shortcoming became apparent – in some specific situations the combination of basic belief masses according to Dempster's rule provided unnatural results. The reason for that is obligatory normalisation of the combined belief masses.

Attempts to overcome that shortcoming have been made, which have resulted in creation of multiple alternative rules for rule combination on the basis of the conjunctive and disjunctive consensus. This paper presents most well-known conjunctive consensus based rules of that kind and provides a comparative analysis of their properties.

2. A short overview of the theory of evidence

The foundations of the theory of evidence were for the first time described by G.Shafer in [2]. Nowadays, an alternative title of that theory, Dempster-Shafer theory, is frequently used. However, in this paper an original name of the theory will be employed.

The main task of the theory of evidence is to extend the limited frames within which probability theory can be correctly applied. Probability theory is based on the concept of the complete group of events $\Omega = \{\omega_i / i = 1, \dots , n\}$. Using up-to-date terminology, events $\omega_i \in \Omega$ will be referred to as *worlds*. *Real world* is an event ω_0, which will occur as a result of experiment or due to certain external conditions. Conceptual requirement of probability theory is that to every event $\omega_i \in \Omega$, the probability of its occurrence, $p(\omega_i)$, $i = 1, \dots , n$, has to be assigned.

In many practical tasks it is impossible to assign all probabilities $p(\omega_i)$ because of the lack of initial information and/or knowledge. The theory of evidence was elaborated to cope with tasks of that kind.

An elementary concept of the theory of evidence is the concept of *frame of discernment* $\Omega = \{\omega_i \ / \ i = 1, \dots, n\}$ - a set of elements (worlds) relevant to the task under consideration. It is postulated that only one element $\omega_0 \in \Omega$ is a real world. Using the initial information, only extents of belief that real world ω_0 belongs to certain subsets (focal elements) $A_j \subseteq \Omega$, can be estimated.

Definition 1. If Ω is a frame of discernment, then function m: $2^\Omega \rightarrow [0, 1]$ is called *basic belief mass assignment* in case if

$$m(\varnothing) = 0;$$
$$\sum_{A \subseteq \Omega} m(A) = 1. \forall A \subseteq \Omega. \quad (1)$$

Let there be a frame of discernment Ω and let basic belief masses be assigned to some subsets (focal elements) of Ω: $m_1(.)$, $m_2(.)$ on the basis of two independent groups of evidences. The combination of these basic belief masses is performed in the theory of evidence according to *Dempster's rule*:

$$m_{12}^D(C) = \frac{\sum\limits_{\substack{i,j \\ A_i \cap B_j \subset C}} m_1(A_i) m_2(B_j)}{1-k}, \quad (2)$$

where k – normalisation constant

$$k = \sum_{\substack{i,j \\ A_i \cap B_j = \varnothing}} m_1(A_i) m_2(B_j). \quad (3)$$

The normalisation constant k is used to normalise the combined belief as a result of which their sum is equal to 1.

Example 1. Assume it is known that the real value of the parameter to be estimated π is within the interval [5, 12] of conditional units: Ω = [5, 12]. Two experts are asked to assign basic belief masses to that the real value of parameter π lies within certain intervals of Ω. Using their knowledge of the subject domain the experts have assigned the intervals and corresponding to them basic belief masses as follows:

Expert A: [6, 8] c $m_A = 0,2$; [7, 9] c $m_A = 0,4$; [8, 10] c $m_A = 0,4$.

Expert B: [5, 8] c $m_B = 0,3$; [7, 10] c $m_B = 0,4$; [9, 12] c $m_B = 0,3$.

These basic belief masses have to be combined according to Dempster's rule. The results of preliminary calculations are given in Table 1.

Table 1

Starting results of basic belief masses combination by experts A and B according to Dempster's rule

Expert B		Expert A					
		Interval	m_A	Interval	m_A	Interval	m_A
Interval	m_B	[6, 8]	0.2	[7,9]	0.4	[8, 10]	0.4
[5, 8]	0.3	[6, 8]	0.06	[7, 8]	0.12	∅	0.12
[7, 10]	0.4	[7, 8]	0.08	[7,9]	0.16	[8, 10]	0.16
[9,12]	0.3	∅	0.06	∅	0.12	[9, 10]	0.12

The boundaries of the resulting intervals are defined as the maximum lower boundary and the minimum upper boundary for the combined intervals.

Let us order the resulting intervals in the order of their lower boundaries increase with the simultaneous summing of the combined belief masses over the same intervals. As a result, we have:

$m_{AB}[6, 8] = 0.06$; $m_{AB}[7, 8] = 0.20$; $m_{AB}[7, 9] = 0.16$; $m_{AB}[8, 10] = 0.16$; $m_{AB}[9, 10] = 0.12$.

Table 1 has three cells corresponding to the empty intersections of the initial intervals. Using that information, let us calculate the value of the normalisation constant: k = 0.06 + 0.12 + 0.12 = 0.30.

Finally, the normalised combined belief masses for the resulting intervals are

$m^D_{AB}[6, 8] = 0.0857$; $m^D_{AB}[7, 8] = 0.2857$; $m^D_{AB}[7, 9] = 0.2286$;
$m^D_{AB}[8, 10] = 0.2286$; $m^D_{AB}[9, 10] = 0.1714$.

Dempster's rule of combination has a strong logical justification, however due to the normalisation operation of the combined belief masses it can provide counter-intuitive results in certain specific situations. It was L.Zadeh who first paid attention to that problem in his work [5]. To better understand the essence of the problem, let us consider the following example.

Example 2. There is a frame of discernment $\Omega = \{A, B, C\}$, whose elements are random events. Two experts have assigned these basic belief masses for the occurrence of those events:

$m_1(A) = 0.99$; $m_1(B) = 0.01$; $m_2(B) = 0.01$; $m_2(C) = 0.99$.

These basic belief masses need to be combined according to Dempster's rule. In this case, a single non-empty intersection of initial focal elements is element B with $m_{12}(B) = m_1(B)m_2(B) = 0.01*0.01 = 0.0001$. The following empty intersections of the initial focal elements have non-zero values of combined belief masses: $m_{12}(A, B) = 0.0099$, $m_{12}(A, C) = 0.9801$, $m_{12}(B, C) = 0.0099$.

The value of normalization constant is k = 0.0099 + 0.9801 + 0.0099 = 0.9999.

Hence, the value of the combined belief mass under consideration is

$$m^D_{12}(B) = 0.0001/(1 – 0.9999) = 1.$$

Let us analyse the result obtained. Both experts assigned very small extents of belief for the possibility of event B occurrence. However, as a result of basic belief masses combination, we have obtained the belief in the occurrence of that event equal to 1. This result reasonably seems to be counter-natural.

Attempts to overcome that essential shortcoming of the theory of evidence have led to the development of alternative rules of combination, which are considered below.

3. Yager's rule of combination

The widely known Yager's rule [4] was historically the first alternative rule of belief combination. Like in the Dempster's rule, the initial combined belief masses are calculated as the orthogonal sum of the basic belief masses

$$q(C) = \sum_{\substack{i,j \\ A_i \cap B_j = C}} m_1(A_i) m_2(B_j). \tag{4}$$

But as apposed to Dempster's rule, here further normalisation of these combined belief masses is not performed. Similarly like in the Dempster's rule, the sum total of the combined belief masses for all empty intersections of the initial focal elements is calculated

$$q(\varnothing) = \sum_{\substack{i,j \\ A_i \cap B_j = \varnothing}} m_1(A_i) m_2(B_j). \tag{5}$$

Finally, the combined belief masses are calculated according to Yager's rule as follows:

$$m^Y(\varnothing) = 0; \tag{6,a}$$

$$m^Y(A) = q(A), \forall A \subset \Omega; \quad (6,b)$$
$$m^Y(\Omega) = q(\Omega) + q(\varnothing). \quad (6,c)$$

From expression (6,c) it can be seen that in Yager's rule of combination the combined belief masses for empty intersections of initial focal elements are ascribed to the frame of discernment, that is to full ignorance.

Example 3. Combine basic belief masses from Example 1 according to Yager's rule. Here, the starting calculations need not to be made. Let us use the results obtained in solving the task in Example 1. It is evident that

$m^Y_{AB}[6, 8] = m_{AB}[6, 8] = 0.06$; $m^Y_{AB}[7, 8] = m_{AB}[7, 8] = 0.20$; $m^Y_{AB}[7, 9] = m_{AB}[7, 9] = 0.16$; $m^Y_{AB}[8, 10] = m_{AB}[8, 10] = 0.16$; $m^Y_{AB}[9, 10] = m_{AB}[9, 10] = 0.12$.

Taking into account the data of Table 1, we have $q(\varnothing) = 0.06 + 0.12 + 0.12 = 0.30$, from which is follows that $m^Y_{AB}(\Omega) = m^Y_{AB}[5, 12] = m_{AB}[5, 12] + q(\Omega) = 0 + 0.30 = 0.30$.

4. Smets' rule of combination

The theory of evidence strictly postulates that the frame of discernment Ω is a set of exhaustive and mutually exclusive elements. In his *model of transferable beliefs* [3], Ph.Smets has abandoned using the postulate of exhaustiveness of the set of elements of the frame of discernment. According to his opinion, some elements can be included in the initial frame of discernment or excluded from it if new evidences provide reasons for that.

Within the transferable belief model, Ph.Smets has formulated this alternative rule of belief combination:

$$m(C) = \sum_{\substack{i,j \\ A_i \cap B_j = C}} m_1(A_i) m_2(B_j), \forall A \subseteq \Omega; \quad (7,a)$$
$$m(\varnothing) = \sum_{\substack{i,j \\ A_i \cap B_j = \varnothing}} m_1(A_i) m_2(B_j). \quad (7,b)$$

Thus, for non-empty intersections of the initial focal elements, the combined belief masses are calculated just as in Dempster's rule, but their normalisation is not done. The sum total of combined belief masses for empty intersections of initial focal elements is referred to the empty set implicitly assuming that parts of that mass can be potentially transferred to those elements which will be further included in the frame of discernment.

Example 4. Combine basic belief masses from example 1, using Smets' rule. Calculations need not to be performed: let us make use of the results obtained in Example 1. We have

$m^S_{AB}[6, 8] = 0.06$; $m^S_{AB}[7, 8] = 0.20$; $m^S_{AB}[7, 9] = 0.16$;
$m^S_{AB}[8, 10] = 0.16$; $m^S_{AB}[9, 10] = 0.12$; $m^S(\varnothing) = 0.30$.

5. Zhang's rule of combination

All the rules of belief combination considered above assume intersecting focal elements as a basis, these rules, however, do not take into account extents of these intersections. An alternative Zhang's rule [6] does account extents of intersection of the corresponding focal elements. If subset C is the result of intersection of subsets A and B, $C = A \cap B$, Zhang introduces the following evaluation of the intersection extent of these subsets:

$$r(A,B) = \frac{|C|}{|A||B|} = \frac{|A \cap B|}{|A||B|}, \quad (8)$$

where $|A|$, $|B|$, $|C|$, $|A \cap B|$ - cardinalities of the corresponding subsets.

Evaluations (8) are calculated for each pair of intersecting initial focal elements. The values of combined belief masses are calculated using Zhang's rule as follows:

$$m^Z(C) = k \sum_{\substack{i,j \\ A_i \cap B_j = C}} \left[r(A_i, B_j) m_1(A_i) m_2(B_j) \right], \qquad (9)$$

where k – normalisation constant;

$r(A_i, B_j)$ are calculated by expression (8).

Example 5. Combine basic belief masses from example 1 using Zhang's rule. The results of the starting combination of basic belief masses can be found in Table 1 (Example 1).

To calculate the evaluations of extents of intersection of relevant intervals, it is necessary to evaluate the cardinality of each pair of initial intervals and the cardinality of their intersection. The cardinalities of relevant intervals will be expressed in terms of their lengths (see Table 2).

Table 2

Lengths of relevant intervals and their intersections

Expert B		Expert A					
		Interval	Length	Interval	Length	Interval	Length
Interval	Length	[6, 8]	2	[7,9]	2	[8, 10]	2
[5, 8]	3	[6, 8]	2	[7, 8]	1	∅	-
[7, 10]	3	[7, 8]	1	[7,9]	2	[8, 10]	2
[9,12]	3	∅	-	∅	-	[9, 10]	1

Now all the data for calculating the values r[.,.] are available. Calculation results are shown in Table 3.

Table 3

Calculated values r[.,.]

Expert B	Expert A					
	Interval	r[.,.]	Interval	r[.,.]	Interval	r[.,.]
Interval	[6, 8]		[7,9]		[8, 10]	
[5, 8]	[6, 8]	0.333	[7, 8]	0.167	∅	-
[7, 10]	[7, 8]	0.167	[7,9]	0.333	[8, 10]	0.333
[9,12]	∅	-	∅	-	[9, 10]	0.167

Let us multiply the values of the combined belief masses for non-empty intersections of marginal intervals in Table 1 by the corresponding values of r[.,.]. Calculation results are presented in Table 4.

Table 4

Combined belief masses scaled using values r[.,.]

Expert B	Expert A					
	Interval	r[.,.]*m_{BC}	Interval	r[.,.]*m_{BC}	Interval	r[.,.]*m_{BC}
Interval	[6, 8]		[7,9]		[8, 10]	
[5, 8]	[6, 8]	0,02	[7, 8]	0,02	∅	-
[7, 10]	[7, 8]	0,013	[7,9]	0,053	[8, 10]	0,053
[9,12]	∅	-	∅	-	[9, 10]	0,02

Upon summing up these scaled values of probability masses by the same combined intervals we have: m'_{AB} [6, 8] = 0.02; m'_{AB} [7, 8] = 0.033; m'_{AB} [7, 9] = 0.053; m'_{AB} [8, 10] = 0.053 and m'_{AB} [9, 10] = 0.02. Sum total of all values m'_{BC} [.,.] is 0.179. Hence the value of normalisation constant is: $k = 1 / 0.179 \approx 5.59$.

By multiplying the values m'_{AB} [.,.] by constant k, finally we have:

$$m^Z_{AB}[6, 8] = 0.112;\ m^Z_{AB}[7, 8] = 0.184;\ m^Z_{AB}[7, 9] = 0.296;\ m^Z_{AB}[8, 10] = 0.296;$$
$$m^Z_{AB}[9, 10] = 0.112.$$

6. A comparative analysis of the examined methods of belief combination

Let us first consider the logic of reasoning underlying each of the rules discussed above. Actually, the starting combination of basic belief masses in all the rules is made similarly as their orthogonal sum. In Dempster's rule, the combined belief masses are normalised to produce 1 if taken as a sum. The idea of such normalisation is connected with that the theory of evidence was historically the first extension of probability theory and due to that it has preserved some features of that theory. The idea of combined belief mass normalisation, with no doubt, is the heritage of probability theory.

The referring of the combined belief masses for non-intersecting initial focal elements to the frame of discernment Ω in Yager's rule has the following logical justification. If basic belief masses assignments are contradicting, their combination cannot diminish the initial uncertainty. So ascribing such belief masses to the frame of discernment is equivalent to relating them to the full ignorance. Contradicting experts' assignments give evidence of their lack of co-ordinated knowledge about the task of, so a specific treatment of such conflicting evaluations in Yager's rule seems to be quite logically justified.

In its turn, the referring of the combined belief masses for non-intersecting initial focal elements to the empty set in Ph.Smets' rule seems to be quite justified within his open world assumption. If we have a potentially non-exhaustive frame of discernment, then conflicting experts' assignments might be the sequence of non-exhaustiveness of exactly that kind. In replenishing the frame of discernment with new elements, the assignments might rather become more co-ordinated.

An undoubtful advantage of Zhang's rule of combination is accounting the extents of intersection of initial focal elements. This is a single rule that accounts that important and unfairly ignored aspect of belief combination.

Which properties should any correct rule of belief combination possess? In [1] the authors state that these features are:

1. The consistence of combination results in all possible cases, more specifically, for any number of groups of evidences, any number of basic assignments of belief masses, any type of frame of discernment and models that might change or remain invariant in time.
2. Commutativity of combination rules.
3. Neutral contribution of the empty belief assignment (when a unitary belief mass is given to the frame of discernment Ω, which gives evidence of full ignorance).

In essence, all rules of combination discussed in this paper possess the above properties.

A very attractive property is the associativity of rules of belief combination. Of the rules presented in this work, only Dempster's rule possesses this property. However, the lack of associativity property is not a principal obstacle for using the rule; this only increases the amount of necessary calculations.

It is impossible to give unambiguous preference to either of the above discussed rules. If the evidences are fully co-ordinated, the best option will be to employ either Dempster's rule of combination, or Yager's rule of combination because both rules will provide identical results. If Yager's rule is employed in the case of conflicting beliefs, one has to take into account the fact that the sum of combined belief masses for intersecting initial focal elements will be less than 1, which might make the interpretation of the obtained results more difficult. However, the relationships between the combined belief masses in this case will be exactly the same as for the case when Dempster's rule is employed. Only amplitudes of the combined belief masses change, while their relative values remain invariant.

Should any doubts concerning the exhaustiveness of the elements of the starting frame of discernment arise, the preference must be given to Ph.Smets' rule of combination. If, due to any reasons, the extents of intersections of the initial focal elements have to be taken into account, Zhang's rule of combination should be used.

In the general case, when a decision about using one or another rule for solving a particular task is made, these factors have to be taken into account:

1. The type of evidences, their quality, independence and safety.
2. The type and properties of the operation of belief combination.
3. The extent of conflict influence on the requested results.

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Užga-Rebrovs Olegs, Kuļešova Gaļina. Pārliecību kombinēšanas alternatīvo likumu salīdzinošā analīze

Liecību teorija ļauj apstrādāt nenoteiktāko informāciju salīdzinājumā ar tradicionālo varbūtību teoriju. Liecību teorijā pārliecību kombinēšanai, kuri tika iegūti uz dažādu liecību grupu pamata, tiek izmantots Dempstera likums. Šim likumam ir stingrs loģisks pamatojums, bet kādos specifiskos apstākļos šis likums var novest pie pretdabiskiem rezultātiem. Mēģinājumi pārvarēt šo būtisko liecību teorijas trūkumu izsauca alternatīvo liecību kombinēšanas likumu rašanos. Šajā darbā tiek apskatīti izplatītākie tāda veida likumi. Pamata atšķirības starp likumiem ir saistītas ar darbībām ar kombinētām pārliecību masām, kuras ir tukšo sākotnēju fokālo elementu krustošanās rezultāts. Darbā tiek prezentētas vispārīgās prasības tāda veida likumiem, kā arī tiek analizētas apskatīto likumu pamata īpašības.

Uzhga-Rebrov Oleg, Kuleshova Galina. A comparative analysis of alternative rules of belief combination

The theory of evidence enables one to process more uncertain information than the conventional probability theory does. To combine beliefs obtained on the basis of different groups of evidences, the Dempster's rule of combination is employed. That rule has a strong logical justification; though in certain specific situations it might yield unnatural results. Attempts to avoid that essential shortcoming of the theory of evidence have resulted in creating alternative rules of belief combination. This paper examines most widespread rules of that kind. The main difference between these rules is in dealing with combined belief masses for empty intersections of the initial focal elements. The paper presents general requirements to the rules of that kind as well as analyses their basic properties.

Ужга-Ребров Олег, Кулешова Галина. Сравнительный анализ правил комбинирования уверенностей

Теория свидетельств позволяет обрабатывать более неопределённую информацию, чем традиционная теория вероятностей. В теории свидетельств для комбинирования уверенностей, полученных на основе различных групп свидетельств, используется правило Демпстера. Это правило имеет строгое логическое обоснование, но в некоторых специфических обстоятельствах оно может дать противоестественные результаты. Попытки преодоления этого существенного недостатка теории свидетельств привели к созданию альтернативных правил комбинирования уверенностей. В настоящей работе рассматриваются наиболее распространённые из таких правил. Основное различие между этими правилами состоит в обращении с комбинированными массами уверенности для пустых пересечений исходных фокальных элементов. В работе представляются общие требования к правилам такого рода, а также анализируются основные свойства рассматриваемых правил.