ISSN 1407-8015 TRANSPORT AND ENGINEERING MAŠĪNZINĀTNE UN TRANSPORTS 2008-8015

TRANSPORT. AVIATION TRANSPORT
TRANSPORTS. AVIĀCIJAS TRANSPORTS

EXTENDED WEAKEST LINK DISTRIBUTION FAMILY. PREDICTION OF STRENGTH-LENGTH DEPENDENCE FOR CARBON FIBERS

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Keywords: carbon fibre, strength, statistical methods, mechanical testing

1. Introduction

This paper presents a development of the models proposed in papers [1, 2]. It is essential also that we process the data already discussed in [3 – 6] revising some conclusions of mentioned papers. According to the “traditional” Weibull model the cumulative distribution function of strength is defined by

$$F(s) = 1 - \exp(-(s/\beta)^{\alpha}). \tag{1}$$

Let the reference length of fiber be equal to l_1 and fiber length equal to L. Then in accordance with “linear law” (LW) model the fiber strength cdf is defined by

$$F(s) = 1 - \exp(-(L/l_1)\ (s/\beta_1)^{\alpha}). \tag{2}$$

It should be noted that here parameter β_1 corresponds to $L = l_1$, β_1 changes if l_1 changes.

The “power law” (PW) Weibull model

$$F(s) = 1 - \exp(-(L/l_1)^{\gamma} (s/\beta_1)^{\alpha}), \tag{3}$$

which has been intensively studied in literature [3 – 5, 7], while providing a much better empirical fit to the strength data of specimens with different length L, lacks the theoretical appeal of the weakest-link models. We derive a new weakest link distribution family (WLDF) based on the assumption of a two-stage failure process. For modeling purposes we consider a specimen (fiber) as a chain of n elements (links) of length l_1. First, the damage process develops along the specimen and defects appear in K elements. Here K is integer random variable, $0 \le K \le n$. Two types of the second stage will be considered in this paper. First type: in every element (containing defect or flaw) the development of fracture process takes place and the strength of the weakest item (link) defines the strength of the specimen. Second type: development of fracture process in crosswise direction takes place only in one, critical element. Then only the probability that the second stage will take place depends on the number of elements but the strength distribution of this element (reflecting the process of accumulation of elementary damages in crosswise direction up to specimen failure) does not depend on this number.

We consider two different versions of the first stage also. First version: defects appear before the loading and their number does not depend on the subsequent loading. Second version: defects appear during loading (instantly or gradually) and their number depends on the load.

2. General description of the model family

2.1. The fracture process takes place in every element

2.1.1. Models of instant fracture. Let K, $0 \le K \le n$, is the number of elements in which defects appear. Let $Y_1, Y_2, ..., Y_K$ be independent random variables which are the strengths of these elements with the same cumulative distribution function (cdf) $F_Y(x)$; $Z_1, Z_2, ..., Z_{n-K}$, $F_Z(x)$ are the same for the elements without defects. It seems reasonable to assume that the random strength of the specimen is the strength of the weakest element

$$X = \min(Y_1, ..., Y_K, Z_1, ..., Z_{n-K}), \tag{4}$$

with the corresponding cdf

$$F(x) = 1 - (1 - F_{Z_{1,n}}(x)) \sum_{k=0}^{n} p_k \delta^k , \tag{5}$$

where

$$\delta(x) = (1 - F_Y(x))/(1 - F_Z(x)), \tag{6}$$

$$F_{Z_{1,n}}(x) = 1 - (1 - F_Z(x))^n . \tag{7}$$

Several different assumptions can be made here. First, let us suppose that the defects appear before loading (technological defects). It can be assumed that the probability of finding a defect in one element, p, is a constant (and a parameter of the model). Then the corresponding binomial probability mass function (pmf) is

$$p_k = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} . \tag{8}$$

If $\lambda = np$ is large enough we can use (as an approximation) the Poisson pmf:

$$p_k = \exp(-\lambda)\lambda^k / k!. \tag{9}$$

In this case the (5) (approximately) can be written in the following way

$$F(x) = 1 - (1 - F_{Z_{1,n}}(x)) \exp(-\lambda(1 - \delta(x))). \tag{10}$$

If initiation of the defects depends on the applied load then it can be assumed that $p = F_0(x)$, where $F_0(x)$ is the cdf of defect initiation stress.

It is worth to note that if $\delta = 1$ then (2) is a particular case of (5) or (10): the Weibull distribution is included in considered WLDF.

2.1.2. Models of gradual accumulation of defects. We consider the process of accumulation of defects as an inhomogeneous finite Markov chain (MC) with finite state space $I = \{i_1, i_2, ..., i_{n+1}, i_{n+2}\}$. MC is in state i_k if there are $(k-1)$ defects, $k = 1,..., n+1$. State i_{n+2} is an absorbing state corresponding to the fracture of specimen. Usually we assume that the Markov chain starts in state i_1 but in the general case the initial distribution is represented by a row vector π given by $\pi = (\pi_1, \pi_2, ..., \pi_{n+1}, \pi_{n+2})$. We further assume that the loading (i.e. the process of nominal stress increase in the specimen cross section) is described by an ascending (up to infinity) sequence $\{x_1, x_2, ..., x_t, ...\}$ and the process of MC state change is described by the transition probabilities matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{23} & \cdots & p_{1(n+1)} & p_{1(n+2)} \\ 0 & p_{22} & p_{23} & p_{24} & \cdots & p_{2(n+1)} & p_{2(n+2)} \\ 0 & 0 & p_{33} & p_{34} & \cdots & p_{3(n+1)} & p_{3(n+2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & p_{(n+1)(n+1)} & p_{(n+1)(n+2)} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

which at the tth-step is a function of x_t, $t = 1, 2,...$. Let the sequence $\{x_t\}$ be fixed, then P is a function of t. Let us note that if $n = \infty$ then the subscript (n+2) is not a number but only a symbol, corresponding to the absorbing state i_{n+2}.

In the new model the number of defects, $K(t)$, and the strength of specimen,

$$X(t) = \min(Y_1, Y_2, ..., Y_{K(t)}, Z_1, Z_2, ..., Z_{n-K(t)}), \tag{11}$$

are random functions of time.

The specimen fracture occurs when the strength of the specimen becomes equal to or less than the current load (stress). Ultimate strength

$$X = x_{T^*}, \tag{12}$$

where

$$T^* = \max(t : X(t) > x_t). \tag{13}$$

Cdf of X is defined on the sequence $\{x_1, x_2, ..., x_m, ...\}$ by

$$F(x_m) = \pi\left(\prod_{j=1}^{m} P(j)\right)u, \tag{14}$$

where $P(j)$ is the transition matrix for step number j, column vector $u = (0, ..., 0, 1)'$ where only the last component is equal to 1 but all the others are equal to 0.

2.1.3. The specimen strength without defect is very large. For the purpose of specification of the models, the general description of which was given in previous section, we additionally have to specify the cdf $F_Y(x)$, $F_0(x)$ (for models with defect number dependence on load), $F_Z(x)$, and, additionally, for 'Markov' models, an initial distribution $\pi = (\pi_1, \pi_2, ..., \pi_{n+1}, \pi_{n+2})$, which, of course, in general case can differ from binomial or Poisson distribution. For 'Markov' models we need to specify also the matrix *P* as function of current stress, x_t, and a sequence $\{x_t\}$ as well.

In this paper we assume that $F_Y(x)$ and $F_0(x)$ are the smallest extreme value (sev) distributions. For the case when location parameter $\theta_0 = 0$ and scale parameter $\theta_1 = 1$ it is assumed that

$$F_Y(x) = 1 - \exp(-\exp(x)), \tag{15}$$

$$F_0(x) = F_Y(x - \delta_0), \tag{16}$$

where in following $x = \log(s)$, s is the strength (expressed in MPa). If $\delta_0 > 0$ then at the same probability of events the stress required for new defect initiation is larger than the stress required for the failure of an element with defect.

For $F_Z(x)$ we consider two assumptions in this paper. First, sev distribution can be assumed again:

$$F_Z(x) = F_Y(x - \delta_Z). \tag{17}$$

Again we can see that if $\delta_Z > 0$ then $F_Z(x) < F_Y(x)$.

But the simplest is the assumption that

$$F_Z(x) = \begin{cases} 0, x < C, \\ 1, x \ge C, \end{cases} \tag{18}$$

where *C* is a very large constant.

Then instead of (4) we have

$$X = \begin{cases} \min(Y_1, ... Y_K), \ K > 0, \\ C, \ \ K = 0. \end{cases} \tag{19}$$

Now Equation (5) can be written in the form

$$F(x) = \begin{cases} 1 - \sum_{k=0}^{n} p_k \delta^k, \ x < C, \\ 1, \ x \ge C, \end{cases} \tag{20}$$

where

$$\delta = 1 - F_Y(x). \tag{21}$$

Then equation (10) takes the following form

$$F(x) = \begin{cases} 1 - \exp(-\lambda F_Y(x)), \ x < C, \\ 1, \ x \ge C. \end{cases} \tag{22}$$

In [1, 2] it was shown that the cdf

$$F(x) = \sum_{k=0}^{\infty} p_k \{1 - (1 - F_Y(x))^{k+1}\} \tag{23}$$

or

$$F(x) = 1 - (1 - F_Y(x))\exp(-\lambda F_Y(x)) \tag{24}$$

where p_k is defined by (7), $\lambda = np$, $p = F_Y(x)$, $F_Y(x)$ is sev cdf, provides a good empirical fit to the strength data of specimens with different length, *L*. Equation (24) can be considered as modification of (8): $F_Y(x)$ is used here instead of $F_{Z_{1,n}}(x)$. And it is not only an approximation of the "binomial" model. Now we can consider the specimen as continuous and define λ by

$$\lambda = \lambda_1 L / l_1 , \tag{25}$$

where *L* is the specimen length, λ_1 is the intensity of defects (the defect number per length l_1; l_1 is some constant). Then function $F_Y(x)$ can be regarded as an element-length-independent cdf of strength distribution in the cross section with a defect, where the number of defective cross sections has the corresponding Poisson distribution.

For Markov models we should specify the matrix *P*. In the case when parameter *C* is very large (the 'theoretical' strength is much higher than the real strength) the probability that in some element a defect appears at the stress x_t under the condition that it has not appeared at the stress x_{t-1} is

$$b(t) = (F_0(x_t) - F_0(x_{(t-1)}))/(1 - F_0(x_{(t-1)})) .$$

If there are *s* defects already the probability that *r* new defects appear, $0 \le r \le k = n - s$, and the total number of defects will be equal to $m = s+r$

$$\widetilde{p}_{sm}(t) = (b(t))^r (1 - b(t))^{k-r} k!/r!(k-r)! .$$

Conditional probability of element fracture at the nominal stress x_t

$$q(t) = (F_Y(x_t) - F_Y(x_{(t-1)}))/(1 - F_Y(x_{(t-1)})) .$$

Corresponding probability that none of the elements fail when there are defects in *m* elements is

$$u_m(t) = (1 - q(t))^m .$$

The probability of coincidence of these events, which we consider as independent, is the probability of transition from state $i = s+1$ to state $j = i+r$

$$p_{ij}(t) = \widetilde{p}_{(i-1)(j-1)}(t) u_{j-1}(t) ,$$

where $i \le j \le (n+1)$.

Conditional fracture probability at state *i*

$$p_{i(n+2)}(t) = 1 - \sum_{j=i}^{n+1} p_{ij}(t) .$$

Of course, $p_{ij}(t) = 0$, if $j < i$, and $p_{(n+2)(n+2)}(t) = 1$.

2.2. The fracture process takes place only in one element

2.2.1. The models of instantaneous failure. In previous models it has been assumed that defects are uniformly distributed along the specimen length. But it is plausible that such uniformity is retained only at the initial stage of loading. More precisely, it can be assumed that upon formation of a weakest link in a chain, the development of failure proceeds only in this link, and the specimen length is of no importance any more. The simplest variant of such a model corresponds to the assumption that the law of strength distribution in the element where this process proceeds (in the cross section where the critical defect is formed) is independent of specimen length, which determines only the probability of formation of an element with defect. The mathematical formulation of this hypothesis is as follows

$$X = \begin{cases} Y, K > 0, \\ Z, K = 0. \end{cases} \tag{26}$$

Here, Y and Z are random variables, which are the strength of element where the failure process proceeds with or without defect, correspondingly.
In this case

$$F(x) = \{1 - (1 - F_0(x))^n\} F_Y(x) + (1 - F_0(x))^n F_Z(x) . \tag{27}$$

If $F_Z(x)$ is defined by (18) then for the case $C = \infty$

$$F(x) = \{1 - (1 - F_0(x))^n\} F_Y(x) . \tag{28}$$

2.2.2. Model of successive formation of at least one defect. The corresponding Markov chain has only three states. The first state corresponds to the absence of defective elements; the second one means the presence of at least one defective element, and the third, absorbing one, means failure of the specimen. The corresponding probabilities at an *t*th step are determined by the formulae

$$p_{11}(t) = [1 - b(t)]^n, \; p_{12}(t) = (1 - p_{11}(t))(1 - q(t)), \; p_{13}(t) = (1 - p_{11}(t))q(t),$$

$$p_{21}(t) = 0, \; p_{22}(t) = 1 - q(t), \; p_{23}(t) = q(t), \; p_{31}(t) = p_{32}(t) = 0, \; p_{33}(t) = 1.$$

In this case

$$X(t) = \begin{cases} Y, K(t) > 0, \\ Z, K(t) = 0. \end{cases} \tag{29}$$

Specification of the cdf and of elements of the matrix *P* (equations for $b(t)$ and $q(t)$) can be made in the same manner as in section 2.1.3.

3. Test data. The processing of test data

3.1. Carbon fiber bundles

We consider the data obtained by Bader and Priest [6] and present the description of these data as given in [4]:

"*There are sixteen samples, consisting of four types of bundles each tested at four different gauge lengths. The four types are (a) single carbon fibres, (b) dry bundles of parallel carbon fibres, (c) impregnated tows of parallel carbon fibres in an epoxy resin matrix, (d) hybrid bundles consisting of tows of carbon embedded in a glass-fibre/epoxy laminate. In each case the failure load under tension was measured in Instron testing machine, and the failure stress computed from that. For types (a)-(c), the tests were repeated independently with fibres of different length. For type (d), the tests were carried out on single specimens of length 200 mm, and these specimens notionally divided up to obtain data for shorter gauge length.*"

Summary (mean values and standard deviations for every sample) of these data is given in [4] in the Table 1. The strength of every specimen for single fibers and for impregnated bundles is given in [3]. We consider the processing of the latter data only.

3.2. Parameter estimation. Linear regression analysis

So we consider the strength data of carbon fibers (four samples with specimen lengths $(L_1, L_2, L_{3,} L_4) = (1,\ 10,\ 20,\ 50\ \text{mm})$, sample sizes $(n_1, n_2, n_3, n_4) = (57,\ 64,\ 70,\ 66)$) and the strength of impregnated bundles (four samples with specimen lengths (L_1, L_2, L_3, L_4) = (20, 50, 150, 300 mm), sample sizes (n_1, n_2, n_3, n_4) = (28, 30, 32, 29)). In [4] the authors consider fitting of these data by LW (equation (1)) and PW (equation (2)) models. However, it appears more important to consider the accuracy of prediction of the strength of fibres with length that differs from the length of the sample used for model parameter estimation.

In the following, we perform the prediction of strength for $L = L_4$ while estimating model parameters using data with $L = L_1$ or with $L = L_1$ and $L = L_2$. The maximum likelihood method is very labour-consuming for the case when cdf is defined by Equation (14) so linear regression analysis (LR) was used for parameter estimation.

Let x_{ij} be *j*th order statistic, $j = 1, 2, ..., \cdots n_i$, n_i is the number of specimens with $L = L_i$, $i = 1, 2, ..., k_L$, k_L is number of different L_i, $E(X_{ij})$ is the expected value of random order statistic X_{ij}, $E(\overset{0}{X}_{ij})$ is the same but for $\theta_0 = 0$ and $\theta_1 = 1$. Then we have the following linear regression model

$$E(X_{ij}) = \theta_0 + \theta_1\, E(\overset{0}{X}_{ij}), \tag{30}$$

where $E(\overset{0}{X}_{ij})$ is a function of $L_{i,}$ n_i and *j*.

Equation (30) can be used for estimation of θ_0 and θ_1 if all the other parameters are fixed. We compare above-mentioned model with both the LW and PW models. If *S* is random strength of specimen with cdf defined by (3) then for $X = \log(S)$ the cdf $F_X(x)$ is defined by

$$F_X(x) = 1 - \exp(-\exp((x - \theta_0)/\theta_1)) \tag{31}$$

with

$$\theta_0 = \log(\beta_1) - (\gamma/\alpha)\log(L/l_1),\ \theta_1 = 1/\alpha\,. \tag{32}$$

So for PW model we have three unknown parameters $\theta_{00} = \log(\beta_1)$, $\theta_{01} = -\gamma/\alpha$ and θ_1

$$E(X_{ij}) = \theta_{00} + \theta_{01} \log(L_i/l_1) + \theta_1\, E(\overset{0}{X}_{ij}) \tag{33}$$

For LW model we have two unknown parameters θ_0 and θ_1

$$E(X_{ij}) = \theta_0 + \theta_1(-\log(L_i/l_1) + E(\overset{0}{X}_{ij})). \tag{34}$$

In (33) and (34) the value of $E(\overset{0}{X}_{ij})$ is the expected value of *j*-th order statistic for sample from sev distribution with sample size n_i, $\theta_0 = 0$ and $\theta_1 = 1$. It is assumed that roughly $E(\overset{0}{X}_{ij}) = \overset{0}{F}{}^{-1}\left(\hat{F}(x_{ij})\right)$, where $\hat{F}(x_{ij}) = (j - 0.3)/(k_l + 0.4)$ is an estimate of $F(x_{ij})$.

As **the measure of data set fitting** the statistic

$$\overline{R}_{LR} = (1 - R^2)^{1/2}, \tag{35}$$

was chosen. Here R^2 is standard statistic of LR analysis (the coefficient of determination).

As nonlinear parameter estimates, the values of the parameters which correspond to the minimum of statistic $\overline{R}_{LR}$ are taken.

As **the measure of the mean prediction error the statistic**

$$Q_1 = \left(\sum_{i=1}^{k_L} (\overline{x}_i - \hat{x}_i)^2 / \sum_{i=1}^{k_L} (\overline{x}_i - \overline{x})^2\right)^{1/2}, \tag{36}$$

but as **the measure of the prediction error** for $L = L_4$ **the statistic** OSPPt-4 (Order Statistic Probability Plot Test statistic, see [2]) were used

$$\text{OSPPt-4} = \left(\sum_{j=1}^{n_4} (x_{4j} - \hat{x}_{4j})^2 / \sum_{j=1}^{n_4} (x_{4j} - \overline{x}_4)^2\right)^{1/2}. \tag{37}$$

Here $\overline{x}_i = \sum_{j=1}^{n_i} x_{ij} / n_i$; predicted $\hat{x}_i = \sum_{i=1}^{n_i} \hat{x}_{ij} / n_i$; predicted $\hat{x}_{ij} = \hat{\theta}_0 + \hat{\theta}_1 E(\overset{o}{X}_{i.j})$; $\hat{\theta}_0$ and $\hat{\theta}_1$ are LR estimates of θ_0 and θ_1, $\overline{x} = \sum_{i=1}^{k_L} \overline{x}_i / k_L$,

3.3. Specifying the models

For the convenience of subsequent reference let us list the full number of specifications and assumptions which define each model in the considered family. We have to specify

1. The conditions under which the initiation of defects takes place. The process of initiation of defects can be a function of **technology** only (p or λ does not depend on load), or this initiation depends on **load** ($p = F_0(s)$ or $\lambda = nF_0(s)$); see (8) and (10).
2. **A distribution of defect number.** The binomial or Poisson distributions are considered as the most appropriate. The defect number cannot exceed the number of elements, so if the finite Markov chains theory is used then the Poisson distribution should be “truncated”.

Remark. We say that a (discrete) distribution is “truncated” in m if instead of discrete rv X we consider the rv

$$Xm = \begin{cases} X, & \text{if } X \le m, \\ m, & \text{if } X > m. \end{cases}$$

In this paper m is equal to the number of elements n.

3. An initial **state distribution,** π, for the models in which Markov chains theory is used. In this paper $\pi = (1,0,...,0)$.
4. CDF of **strength of elements without defects**, F_Z, the cdf of **strength of elements with defects**, F_Y, and the cdf of **defect initiation stress,** F_0 (if the process of defect initiation is assumed to be a function of load).
5. The **sequence of loads** (stresses) $\{x_t\}$ if the finite Markov chains theory is used. In this paper, as a rule, $\{x_t\}$ is a sequence of numbers uniformly distributed in some interval.

There is a great deal of variations of the models from WLDF. In this paper we consider four basic models with $F_Y(x)$, $F_0(x)$ and $F_Z(x)$ defined by (15 – 18), with binomial (for "Markov's" model) or Poisson distribution of defect number :

A. The structure of the process is defined by (4); the cdf is specified by (6 – 10).

B. The structure of the process is defined by (26); the cdf is defined by (28).

MA. The structure of the process is defined by (11-13); the cdf is defined by (14), the matrix P is described in 2.1.2.

MB. The structure of the process is defined by (29); cdf is defined by (14), the matrix P is described in 2.2.2.

Simultaneously the calculation for both LW and PW (if possible) was done.

For prediction stability it is very important to minimize the number of unknown parameters. For this purpose in this paper for all models and materials we set $l_1 = L_1$.

For basic model A we have considered four versions:

1. AL with $\lambda = nF_0(x)$; $F_Z(x)$ is defined by (17).
2. AT with $\lambda = n\lambda_1$, where λ_1 is technological defect intensity for $L = L_1$; $F_Z(x)$ is defined by (17).
3. ALmod with cdf defined by (24) with $\lambda = nF_0(x)$;
4. ATmod with cdf defined by (24) with $\lambda = n\lambda_1$, where λ_1 is technological defect intensity for $L = L_1$.

Note that ALmod model (PS model in[1], p-sev-sev in [2]) corresponds to the following substitution:

$F_Y(x) = 1 - \exp(-\exp(x))$ is used instead of $F_{Z_{1,n}}(x) = 1 - \exp(-n\exp(x - \delta_Z))$.

3.4. Comparison of the models when parameters are estimated from the sample $L = L_1$

For this case only the models with two unknown parameters can be used. So we cannot use PW model but when for model A we set δ_0 and δ_z are equal to 0 then this model coincides with the LW model. From the remaining models for **carbon fiber data** the MB-model provides the minimum of $\overline{R}_{LR}$ = 0.0866 (for LW model this value is equal to 0.1135). But LW model provides much better prediction for $L = L_4$: OSPPt-4 = 0.1953. The model MA provides nearly the same result (see Fig. 1): OSPPt-4 = 0.2292 (with somewhat better fitting: $\overline{R}_{LR}$ = 0.091). The prediction of the model MB for $L = L_4$ is very poor: OSPPt-4 = 2.0494.

The model B provides the minimum of $\overline{R}_{LR}$ = 0.1717 and the best prediction for $L = L_2$ and $L = L_3$ for bundles of 1000 impregnated carbon fibers. The model LW fits the experimental data somewhat worse: $\overline{R}_{LR}$ = 0.213 but this model provides better prediction OSPPt-4 (OSPPt-4 = 0.4412) (see Fig. 2). The model MA provides OSPPt-4 = 0.5375 (see Fig. 3).

3.5. Comparison of the models when parameters are estimated from two samples, with $L = L_1$ *and* $L = L_2$

This time we can compare all 7 models. The best fitting of **single carbon fiber** experimental data is provided by model PW: $\overline{R}_{LR}$ = 0.1705. Close result is yielded by the model MA: $\overline{R}_{LR}$ = 0.1737 (see Fig. 4). For model LW $\overline{R}_{LR}$ = 0.1803 but it provides the best prediction: OSPPt-4 = 0.2769. Model PW performs somewhat worse: OSPPt-4 = 0.4026. It is worth mentioning again that model A coincides with the model LW if δ_0 and δ_z are equal to 0. The attempt to minimize $\overline{R}_{LR}$ using different values of δ_z (at $\delta_0 = 0$) has was not successful.

For impregnated bundle data PW model provides the best fitting: $\overline{R}_{LR} = 0.2109$ (with OSPPt-4 = 1.1647). LW model provides best prediction (see Fig. 5): OSPPt-4 = 0.5386 (with $\overline{R}_{LR} = 0.3680$). MA model provides similar results: OSPPt-4 = 0.6981 (with $\overline{R}_{LR} = 0.4218$). MB model provides some better fitting ($\overline{R}_{LR} = 0.2845$) but worse prediction (OSPPt-4 = 0.8835). But there is suspicion that the test of impregnated bundles was not faultless.

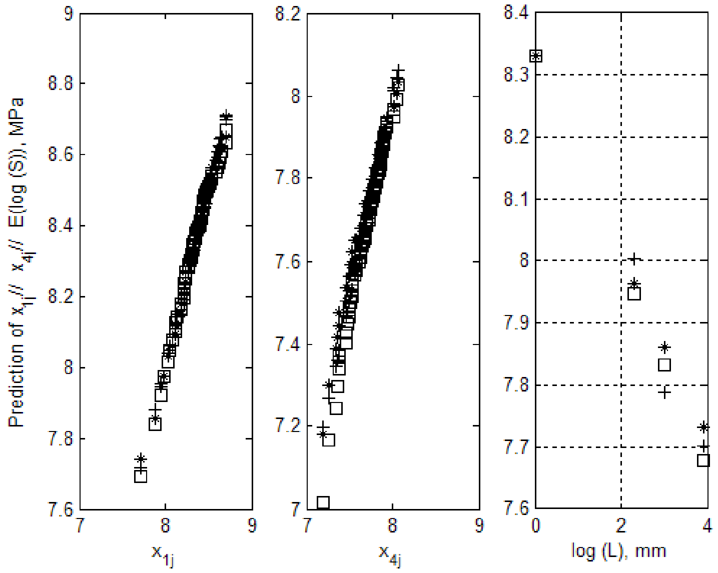

Fig. 1. Single fiber test data (+); predictions $\hat{x}_{1j}, \hat{x}_{4j}$ and $\overline{x}_j$ using MA-model (*) ($\overline{R}_{LR} = 0.091$, OSPPt-4 = 0.2292, $Q_1 = 0.1833$) and LW model (□) ($\overline{R}_{LR} = 0.1135$, OSPPt-4 = 0.1953, $Q_1 = 0.1526$). Initial data – sample with $L = L_1$

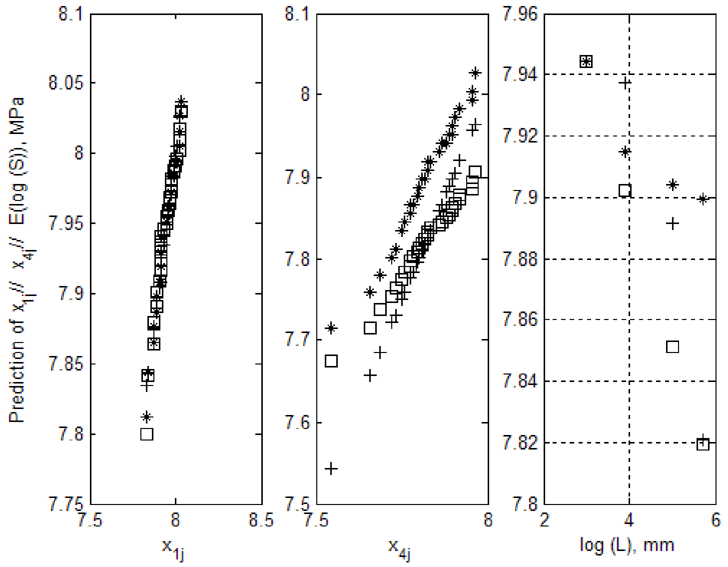

Fig. 2. Data of bundles of 1000 impregnated carbon fibers (+); predictions $\hat{x}_{1j}, \hat{x}_{4j}$ and $\overline{x}_j$ using B-model (*) ($\overline{R}_{LR} = 0.1717$, OSPPt-4 = 0.8568, $Q_1 = 0.8381$) and LW model (□) ($\overline{R}_{LR} = 0.2130$, OSPPt-4 = 0.4412, $Q_1 = 0.5436$). Initial data – sample with $L = L_1$

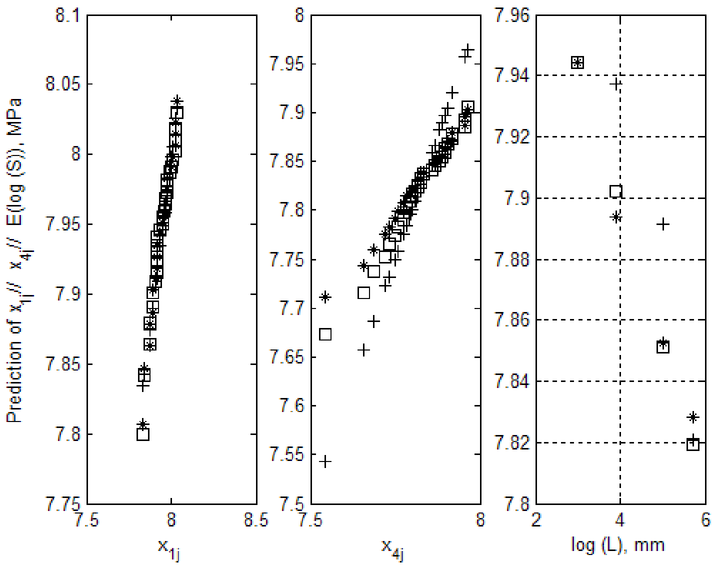

Fig. 3. Data of bundles of 1000 impregnated carbon fibers (+); predictions $\hat{x}_{1j}$, $\hat{x}_{4j}$ and $\bar{x}_j$ using MA-model (*) ($\overline{R}_{LR}$ = 0.1793, OSPPt-4 = 0.5375, Q_1 = 0.4283) and LW model (□) ($\overline{R}_{LR}$ = 0.2130, OSPPt-4 = 0.4412, Q_1 = 0.5436). Initial data – sample with $L = L_1$

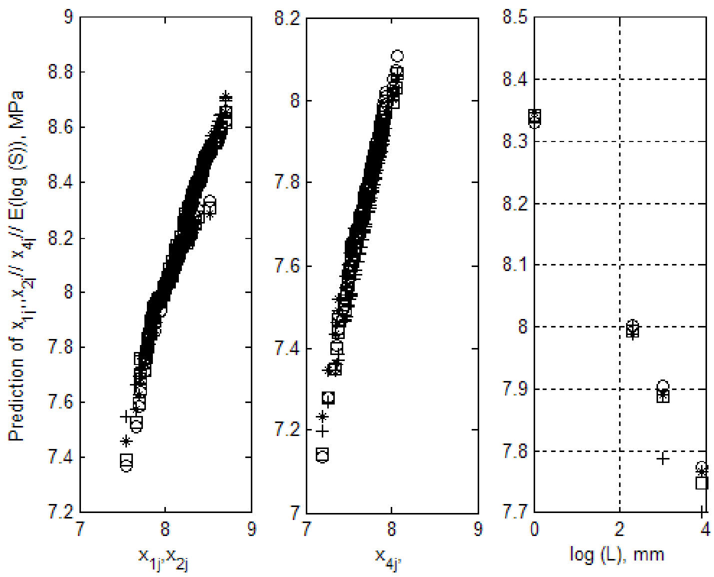

Fig. 4. Single carbon fiber test data (+) ; predictions $\hat{x}_{1j}$, $\hat{x}_{2j}$;$\hat{x}_{4j}$; $\bar{x}_j$ using MA-model (*) ($\overline{R}_{LR}$ = 0.1737, OSPPt-4 = 0.3863, Q_1 = 0.2549), LW model (□) ($\overline{R}_{LR}$ = 0.1803, OSPPt-4 = 0.278, Q_1 = 0.2268) and PW model (o) ($\overline{R}_{LR}$ = 0.1705, OSPPt-4 = 0.4026, Q_1 = 0.2809). Initial data – samples with $L = L_1$ and $L = L_2$.

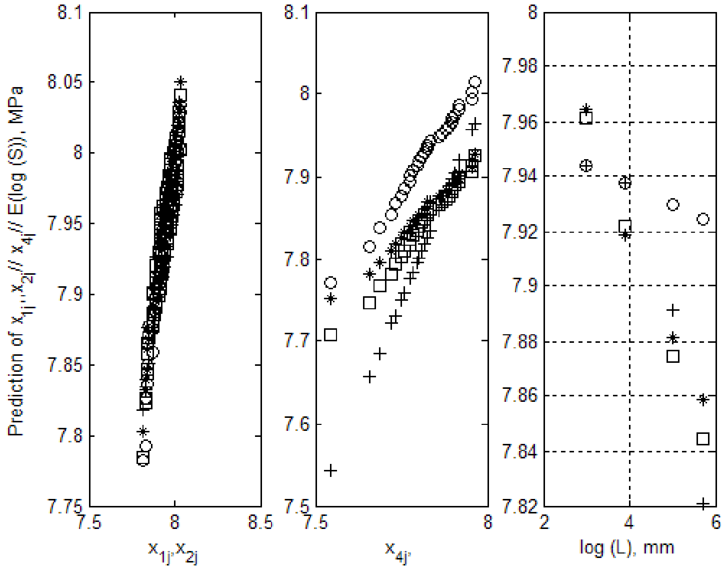

Fig. 5. Data of bundles of 1000 impregnated carbon fibers (+); predictions $\hat{x}_{1j}$, $\hat{x}_{2j}$; $\hat{x}_{4j}$; $\bar{x}_j$ using MA-model (*) ($\overline{R}_{LR}$ = 0.4218, OSPPt-4 = 0.6981, Q_1 = 0.4890), LW model (□) ($\overline{R}_{LR}$ = 0.368, OSPPt-4 = 0.5386, Q_1 = 0.4890) and PW model (o) ($\overline{R}_{LR}$ = 0.2109, OSPPt-4 = 1.1647, Q_1 = 1.1228). Initial data – samples with $L = L_1$ and $L = L_2$

Summary

A new description of extended WLDF is offered. As distinct from our previous publications, the strength of items without defects is taken into account; two types of defects ("technological", i.e. independent on load and dependent on load) and two types of the influence of defect number on the specimen strength are introduced**.** In previous publications it was assumed that "theoretical" strength was equal to infinity. A more general case is discussed in this paper.

We have considered four basic models from WLDF which are defined by the following four structures:

A. $X = \min(Y_1,...,Y_K, Z_1,...,Z_{n-K})$; B. $X = \begin{cases} Y, K > 0, \\ Z, K = 0; \end{cases}$

MA. $X(t) = \min(Y_1, Y_2,..., Y_{K(t)}, Z_1, Z_2,..., Z_{n-K(t)})$; MB. $X(t) = \begin{cases} Y, K(t) > 0, \\ Z, K(t) = 0. \end{cases}$

Here K has binomial, $b(k;n,p)$, (for finite n), or ("truncated" for finite "Markov's" models) Poisson pmf, $p(k;\lambda)$, (if $n = \infty$).

Parameters p and λ can be independent of load ("technological" defects) or $p = F_0(x)$, $\lambda = nF_0(x)$, where $F_0(x)$ is some cdf of defect initiation stress. The random process $K(t)$ is defined as a Markov chain.

In this paper for numerical examples of prediction of strength-length dependence we have used the data set of single carbon fibers and of impregnated bundles.

The comparison with LW or, if possible (if for parameter estimation there are at least two samples with different lengths), with both LW and PW models was made for every considered model from WLDF. It turned out that there is no best model for each material and each set of length, $\{L_1, L_2,..., L_k\}$. For single fiber data, LW model is appropriate, but MA model gives the most stable prediction nearly in all considered examples. If it is applicable, the PW model provides, as a rule, the best fitting of test data but not necessarily the best prediction (see Fig. 5). For considered data, as a rule, LW or MA models provide the best prediction. It is an unexpected result, because in [4] and [7] PW model was suggested as being superior to LW model. However, note that PW parameter for single carbon fiber data is $\gamma = 0.90$ [3], i.e. quite close to the limiting value of $\gamma = 1$ at which PW and LW distributions coincide. This is likely to be the reason of the relative closeness of LW and PW model performance, while a more dramatic deviation of size effect from LW (e.g. $\gamma = 0.58$ for glass [8] and $\gamma = 0.46$ for flax fibers [9]) could also allow to better discriminate the predictive capacity of WLDF models.

It seems reasonable that PW model provides better fitting of experimental data than LW model mainly because it has an additional parameter (in comparison with LW model) and a very universal Taylor's-series-type (see Equation (33)) description of strength-length dependence. Obviously, much better fitting can be obtained if, for example, the number of terms in Equation (33) is increased. The PW model is widely used to describe experimental data, but, in our opinion, its theoretical substantiation (for example, in [4]) is not convincing enough. The models from WLDF preserve the main idea of the weakest-link model upon changes in the specimen length. It seems that this family has great potential (for example, there is a wide choice of $F_Z(x), F_Y(x), F_0(x), \ldots$) and deserves to be studied much more thoroughly using much more test data.

We should mention also that the considered distribution family can be applied not only to the fiber strength analysis but to analysis of reliability of any series system with two types of elements.

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Paramonovs Ju., Andersons J. Vājākā posma varbūtības sadalījumu kopas paplašinājums. Oglekļa šķiedras stiprības atkarības no garuma prognozēšana

Vājākā posma varbūtības sadalījumu kopa (VLSK) tika aprakstīta autoru darbos. Šajā rakstā ir izstrādāts šīs kopas paplašinājums un apstrādāti dati par stiprības atkarību no oglekļa šķiedras garuma. Atšķirībā no iepriekšējiem pētījumiem, tika ievērota arī elementu bez sākotnējiem defektiem stiprība. Aplūkoti divi defektu tipi – tie, kas parādās neatkarīgi no slodzes („tehnoloģiskie"), un tie, kā parādās slogošanas procesā, kā arī divi veidi, kādos defektu skaits ietekmē parauga stiprību. Modeļu parametru vērtējumi veikti, izmantojot Badera un Priesta eksperimentālos datus, kuri jau agrāk tika izmantoti vairākos rakstos un minēti literatūras sarakstā. Tika

apstrādātas astoņas statistiskās izlases, kas attiecināmas uz divu veidu eksperimentālu paraugu izmēģinājumiem: brīvās oglekļa šķiedras un 1000 šķiedru kūļi, kuri bija impregnēti ar epoksīdsveķiem. Katra veida paraugi bija izmēģināti pie četriem dažādiem garumiem. Katra gadījumā sagrūšanas slodze tika fiksēta ar pārbaužu iekārtas Instron mērīšanas sistēmu. Parādīts, ka daži VLSK modeļi un tradicionālais lineārais Veibula modelis nodrošina precīzāku oglekļa šķiedras stiprības atkarības no garuma prognozi nekā pakāpes likuma Veibula modelis, kurš, savukārt, dažreiz tomēr sniedz vislabāko visu eksperimentālo datu kopuma aprakstu.

Paramonov Yu., Andersons J. Extended weakest link distribution family. Prediction of strength-length dependence for carbon fibers

An extension of the weakest link distribution family (WLDF), developed in [1, 2], and its application to the carbon fiber strength-length data are presented in this paper. As distinct from our previous publications, the strength of items without defects is taken into account; two types of defects (independent on load (“technological”) and dependent on load) and two types of the influence of defect number on the specimen strength are considered.Estimation of the model parameters was made using the data obtained by Bader and Priest which was already discussed in referenced papers. Results of testing of eight samples, consisting of two types of specimens each tested at four different gauge lengths, was studied. The two types are (a) single carbon fibres, (b) bundles of 1000 parallel carbon fibers impregnated in an epoxy resin matrix. In each case the failure load under tension was measured in Instron testing machine. It is shown that for carbon fiber material some models from WLDF and linear law (LW) traditional Weibull model provide a better prediction of strength-length dependence than power law Weibull model which sometimes provides better fitting of experimental data.

Парамонов Ю., Андерсонс Я. Расширенное семейство распределений слабейшего звена. Предсказание прочность-длина зависимости для углеродных волокон

Дано описание расширенного семейства распределений слабейшего звена (СРСЗ), ранее представленного в работах [1, 2] и результатов его использования для обработки данных о зависимости прочности от длины углеродных волокон. В отличие от предшествующих публикаций принята во внимание и прочность элементов, в которых отсутствуют начальные повреждения. Рассмотрено два типа дефектов (появляющихся независимо от процесса нагружения (технологические дефекты) и появляющихся в процессе нагружения) и два типа влияния числа дефектов на зависимость прочности образца от длины. Оценки параметров моделей были сделаны, используя данные, полученные Бадером и Приестом, уже обсуждавшиеся в некоторых статьях, упомянутых в приведенном списке использованных литературных источников. Были проанализированы восемь статистических выборок, относящихся к испытаниям двух типов образцов: свободные углеродные волокна и пучки из 1000 волокон, пропитанных эпоксидной смолой. Каждый тип образцов был испытан при четырёх различных длинах. В каждом случае разрушающая нагрузка замерялась измерительной системой испытательной машины типа Инстрон. Показано, что некоторые модели из СРСЗ и традиционная линейная модель Вейбулла дают лучшее прогнозирование зависимости прочность-длина, нежели степенная Вейбулловская модель, которая, однако, иногда даёт наилучшее описание (“выравнивание”) экспериментальных данных.