

About the New Periodic Solutions of the Mathieu-Duffing Equation in the Principal Zone of Instability

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Abstract - The purpose of the article is to present new solutions of the Mathieu-Duffing equation in the zone of the principal parametric resonance – the stationary solutions with the frequency equal to the frequency of parametric excitation. These solutions are obtained by combining numerical and analytical research methods, but their verification is done by direct integration of the original equation. Initially monoharmonic analysis of the solutions under study has been performed, and this analysis gave contradictory and inconsistent results. Only further polyharmonic approach has made it possible to do definite conclusions.

Keywords - dynamical system, Mathieu-Duffing equation, stationary solution, numerical analysis

I. INTRODUCTION

The behavior of nonlinear dynamical systems due to its variety is under the constant attention of researchers. As it was shown by V.V. Bolotin [1], oscillations on the bounds of the zone of the principal parametric resonance are executed with frequency, which is two times smaller in comparison with the parametric excitation frequency. Due to the nature of oscillations on the bounds of the parametric instability zone, it is assumed by many researchers that inside this zone the halving of the excitation frequency is occurred, too. The later discovery of chaotic motions in such systems did not change these concepts (at last for small values of parameters of the system) [2]. The review of research on nonlinear dynamics, which is given in [3], shows the increasing interest to the analysis of systems with periodic coefficients. But there are not found results relating to the conditional "average" values of the parameters of the system. However, the study of the systems with piecewise linear elastic characteristics, performed with the use of analog computers [4], showed the possibility of dividing the frequency of excitation in the zone of the principal parametric resonance not only in two, but in a different value. Looking for additional clarification on this matter the Mathieu equation with a symmetric characteristic of the Duffing is considered below, and the conditions of the existence of stationary solutions in the zone of the principal parametric resonance with a period equal to the period of parametric excitation are studied.

II. EQUATION UNDER STUDY

The Mathieu-Duffing equation is considered in the form

$$\ddot{x} + \beta \dot{x} + \omega_0^2 (1 - 2\mu \cos 2\omega t)x + \delta x^3 = 0, \quad (1)$$

where parameters β, μ, δ are assumed as positive. This equation describes the dynamics of the system with harmonic parametric excitation and nonlinear cubic characteristic of restoring force. Equation (1) obviously has a trivial solution, but under certain relations between the coefficients of excitation μ and resistance β in certain frequency ranges the rest state becomes unstable and parametric oscillations are excited. Zones of instability have wedge-shaped areas in the plane of parameters $\left(\mu, \frac{\omega_0}{\omega}\right)$. The most wide zone of instability is the principal one, which is contiguous with the point $\frac{\omega_0}{\omega} = 1$ when $\beta = 0$. The peculiar feature of these oscillations lies in the fact that their amplitudes grow exponentially, and even the existence of a linear viscous damping is not enough for the limitation of amplitudes of oscillations, if $\delta = 0$. However parametric oscillations have limited amplitudes in the case $\delta \neq 0$, if there are nonlinearities in the system.

III. MONOHARMONIC ANALYSIS OF SOLUTIONS

For analytical research the method of harmonic balance is used. Considering the oscillations in the main zone of instability and assuming $\omega_0^2 = 1$ and $\omega = 1$, the monoharmonic solution of the equation (1) can be sought in the form of

$$x_{an}(t) = A_0 + A_1 \cos 2t + B_1 \sin 2t, \quad (2)$$

where A_0, A_1, B_1 are the constant coefficients. After substituting of solution (2) into the equation (1) and some manipulations, a system of algebraic equations for determination of the coefficients A_0, A_1 and B_1 can be written in the following form:

$$\begin{cases} A_0 - \mu A_1 + \delta \left(A_0^3 + \frac{3}{2} A_0 A_1^2 + \frac{3}{2} A_0 B_1^2 \right) = 0, \\ -3A_1 + 2\beta B_1 - 2\mu A_0 + \\ \quad + \delta \left(3A_0^2 A_1 + \frac{3}{4} A_1^3 + \frac{3}{4} A_1 B_1^2 \right) = 0, \\ -3B_1 - 2\beta A_1 + \\ \quad + \delta \left(3A_0^2 B_1 + \frac{3}{4} A_1^2 B_1 + \frac{3}{4} B_1^3 \right) = 0. \end{cases} \quad (3)$$

With the parameter values $\beta = 0.001$, $\mu = 0.3$ and $\delta = 1/3$ the solution of system (3) was conducted in MATLAB using the procedure **solve**. As the result the following solutions of the equation (1) were found:

$$\begin{aligned} x_{an.1,2} &= \pm(0.1478 + 3.4556\cos 2t + 0.2710\sin 2t), \\ x_{an.3,4} &= \pm(0.0116 + 0.2703\cos 2t + 3.4536\sin 2t). \end{aligned} \quad (4)$$

The existence of the first pair of solutions (4) has been confirmed by the direct numerical integration of equation (1). But the existence of solutions $x_{an.3,4}$ did not receive numerical confirmation (see Fig. 1). The further calculation of multipliers $\rho_{1,2}$ of the variational system corresponding to

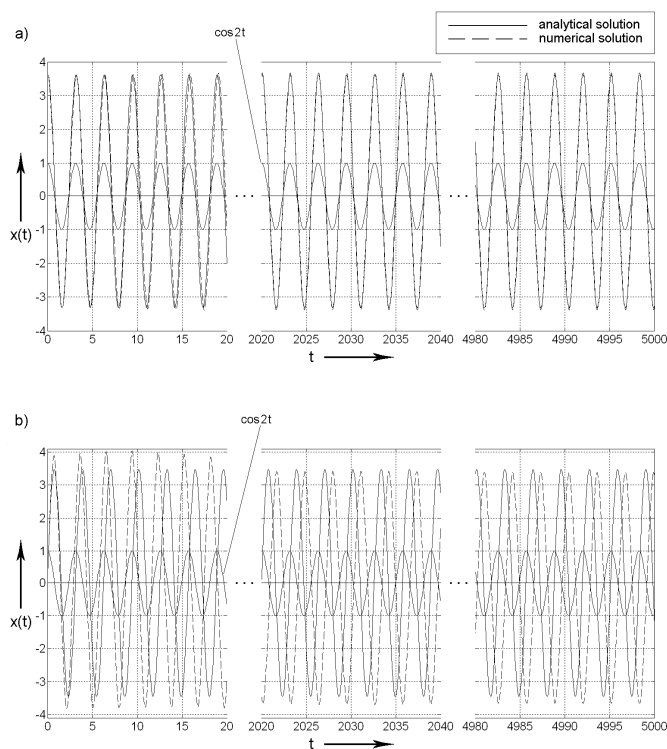


Fig. 1. The solutions of the equation (1): a) numerical $x(t)$ and analytical $x_{an.1}(t)$; b) numerical $x(t)$ and analytical $x_{an.3}(t)$

this pair of solutions has shown, that $\rho_{1,2} = 0.8167 \pm 0.5759i$, i.e. $|\rho_{1,2}| = 0.9987 < 1$, and, therefore, solutions $x_{an.3,4}$ are stable. Thus the apparent contradiction between the results of the analytical calculation (4) and its numerical presentation was observed (see Fig. 1). The further exhaustion of ODE solvers available in the library of MATLAB, as well as the variation of control parameters of this contradiction did not eliminate this contradiction.

The hypothesis, which can explain the observing contradiction between the numerical and analytical results, has been formulated in the following form: the oscillations of the kind $x_{an.3,4}$ exist, but despite their monoharmonic character the system of equations (3) for determination of coefficients A_0 , A_1 and B_1 , compiled on the basis of monoharmonic expression (2), is a crude, and as a result the values of coefficients A_0 , A_1 , B_1 are only the approximate ones. Therefore the corresponding equation in variations and calculation of its multipliers can be inadequate.

If this assumption is true, then detailed and consistent results can be expected with greater consideration of harmonic components in the desired solutions.

IV. POLYHARMONIC ANALYSIS OF THE SOLUTIONS

The equation (1) has a nonlinearity of the polynomial type. Taking account of the nature of the transformations and the desired algorithmization of this process, it is expedient to use the complex form of Fourier series. As before, solving of the equation (1) is made with the method of harmonic balance, but solution is sought in the form of

$$x_{an}(t) = \sum_{n=-N}^N c_n e^{i\omega_1 n t}, \quad (5)$$

where N is the number of harmonics taken into account in the expansion.

The relationship of complex conjugate coefficients c_n and c_{-n} with coefficients A_n and B_n at $\cos n\omega_1 t$, $\sin n\omega_1 t$ of usual Fourier series is determined by the expressions

$$A_n = c_n + c_{-n}, \quad B_n = (c_n - c_{-n})i. \quad (6)$$

By the substituting (5) in (1) it can be obtained:

$$\begin{aligned} & \sum_{p=-N}^N \left(\omega_0^2 + i\beta p\omega_1 - p^2\omega_1 \right) c_p e^{ip\omega_1 t} - \\ & - \mu\omega_0^2 \sum_{p=-N}^N c_p e^{i(p\omega_1 + 2\omega)t} - \\ & - \mu\omega_0^2 \sum_{p=-N}^N c_p e^{i(p\omega_1 - 2\omega)t} + \\ & + \delta \sum_{p=-3N}^{3N} \left(\sum_{n=-N}^N \sum_{k=-N}^N c_n c_k c_{p-n-k} \right) e^{ip\omega_1 t} = 0, \end{aligned} \quad (7)$$

where the condition $-N \leq p - n - k \leq N$ is valid. The value of the frequency ω_1 of fundamental harmonic is assigned taking account of the anticipated nature of the regime under study. Then equating coefficients of equal powers of $e^{i\omega_1 t}$, the system of the equations for determining the coefficients c_n is obtained.

Focusing on the middle of the principal zone of parametric instability, it is assumed $\omega_0^2 = 1$ and $2\omega = 2\omega_0 = 2$ (i.e. $\omega = 1$) in expressions (7). But the fundamental frequency ω_1 of the oscillations is considered equal to the frequency of the parametric excitation, i.e. $\omega_1 = 2\omega$. By equating coefficients, the following system of equations for determination of c_n is obtained:

$$\begin{cases} (1 - 4N^2 - i2\beta N)c_{-N} - \mu c_{-N+1} + \delta f_{-N} = 0, \\ (1 - 4p^2 + i2\beta p)c_p - \mu c_{p-1} - \mu c_{p+1} + \delta f_p = 0, \\ p = \overline{-N+1, N-1}, \\ (1 - 4N^2 + i2\beta N)c_N - \mu c_{N-1} + \delta f_N = 0, \end{cases} \quad (8)$$

where
$$f_p = \sum_{n=-N}^N \sum_{k=-N}^N c_n c_k c_{p-n-k} \cdot$$

Further analysis of equations (8) is made using package Maple. Necessary initial approximations were determined using $x_{an,3}(t)$. As a result, there were certain trends in the progressive increase of number N of harmonic components in the decomposition (5). Namely, there is a marked refinement of lower coefficients A_0, A_1, B_1 up to 40%, while the amplitude of the higher harmonic components decreases with the increasing of the number N and gradually "disappears". As an example, the refined solutions $x_{an,3,4}$ obtained for $N = 10$ by transforming them into the usual trigonometric form, using the equation (6), are presented (the expansion coefficients are indicated with the accuracy of up to five digits after the decimal point):

$$\begin{aligned} x_{an,3,4}(t) = & \pm(0.00884 + \\ & + 0.19389 \cos 2t + 3.39108 \sin 2t - \\ & - 0.01156 \cos 4t - 0.10594 \sin 4t - \\ & - 0.01887 \cos 6t - 0.10917 \sin 6t + \\ & + 0.00142 \cos 8t + 0.00612 \sin 8t + \\ & + 0.00096 \cos 10t + 0.00327 \sin 10t - \\ & - 0.00010 \cos 12t - 0.00028 \sin 12t - \\ & - 0.00004 \cos 14t - 0.00009 \sin 14t + \\ & + 0.00001 \cos 16t + 0.00001 \sin 16t). \end{aligned}$$

Thus, the nature of the solution remains practically monoharmonic, but the amplitudes of the main components are significantly corrected. A higher level of agreement is observed in this case and with the numerical results. Fig. 2 provides the solution $x_{an,3}(t)$ for different values N and the results of direct numerical integration of the equation (1). Rather high correlation between the decisions at the initial stage of the motion (in the case of $N = 3, N = 10$), in our opinion, confirms the existence of the oscillations of this type and provides the basis for investigation of their stability.

Refinements achieved in the solutions $x_{an,3,4}$ caused the relevant revisions in the values of the multipliers, too. Their re-calculation to clarify the variational equations for $N = 10$ showed, that $\rho_1 = 1.3510$ and $\rho_2 = 0.7394$. Thus, because of $|\rho_1| > 1$ the solutions $x_{an,3,4}$ are still unstable, which is consistent with the behavior of the numerical solutions (see Fig. 2). The location of the multipliers relative to the unit circle in the complex plane is shown in Fig. 3.

The presented images demonstrate the consistency of the obtained results for $N = 3$ and $N = 10$. On the one hand, these graphs show a gradual withdrawal of the numerical solutions from the analytical ones, but on the other hand this fact can be explained by the instability of these solutions.

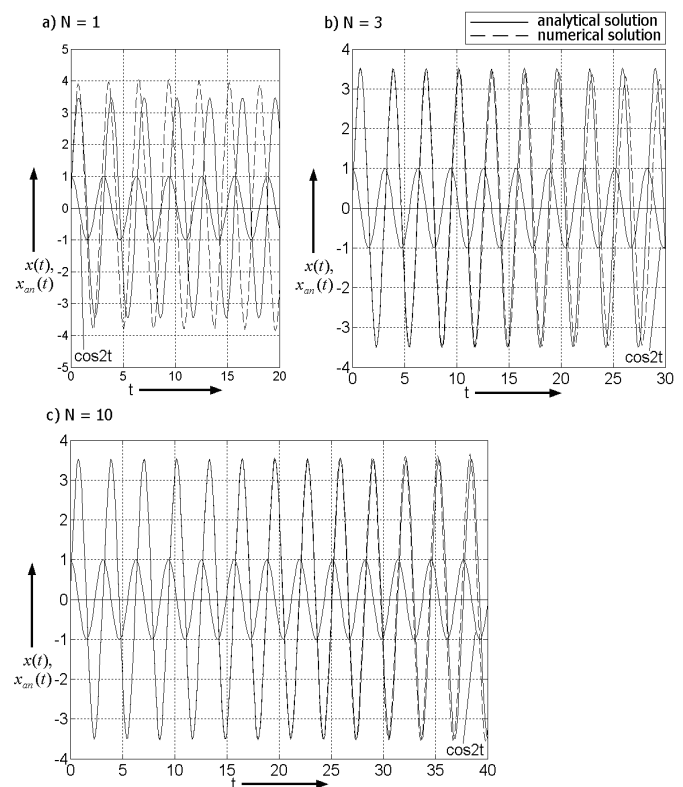


Fig. 2. Numerical $x(t)$ and analytical $x_{an,3}(t)$ solutions of the equation (1) for different values N

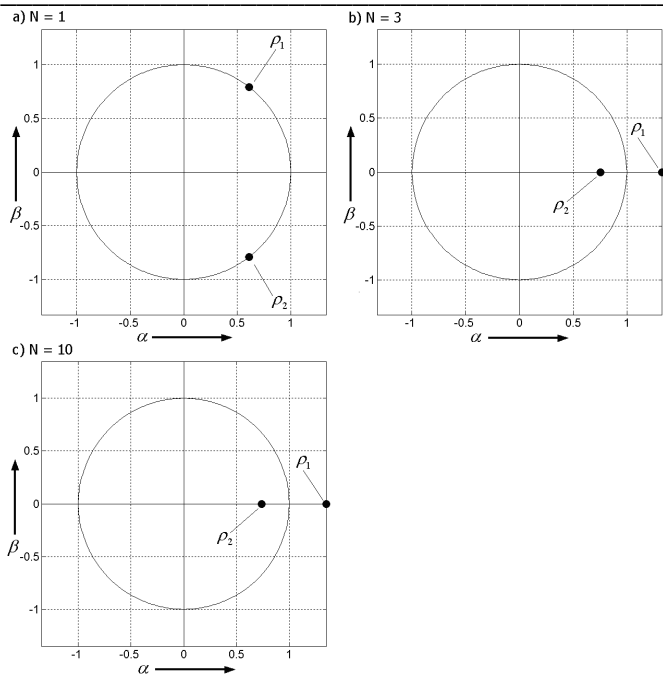


Fig. 3. The multipliers $\rho = \alpha + i\beta$ for different values N

V. THE “MECHANISM” OF EXCITATION OF THE NEW OSCILLATIONS

As is known [5], the zones of instability of the trivial solution of the Mathieu equation (for $\delta = 0$ in the equation (1)) are similar to the wedge-shaped areas on the plane (μ, ω) . In the absence of damping these zones lie near the frequencies $\omega = \omega_0/n$ (where $n = 1, 2, 3, \dots$). Besides, on the borders of even zones the oscillation period coincides with the period of excitation. Further, it is known, that in non-linear systems with hard elasticity, the amplitude-

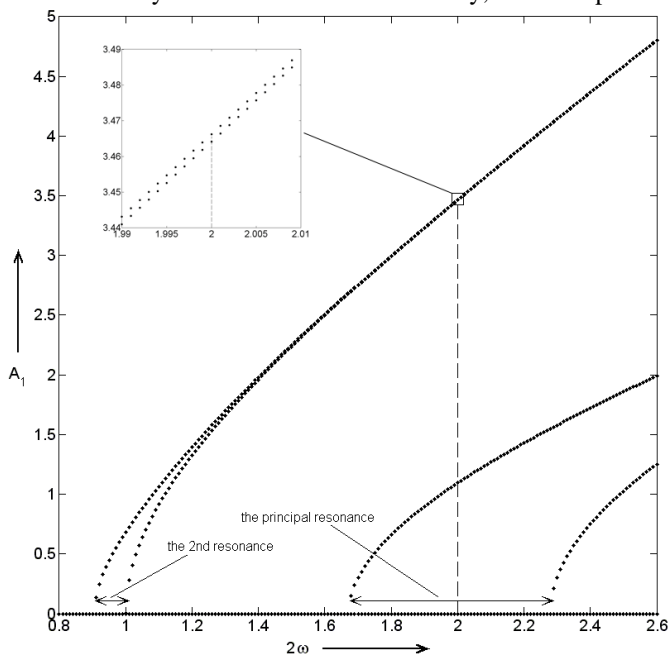


Fig. 4. Amplitude-frequency characteristics of the stationary oscillations corresponding to the first two resonances

frequency characteristics (AFC) are sloped towards higher frequencies. Therefore, in order to explain the causes of the existence of the found solutions, the AFC of the basic response regimes for the two first zones of parametric resonance have been plotted.

Systems of equations to determine c_n were formed on the basis of (7), while for the principal parametric resonance it is supposed $\omega_1 = \omega$ and for the 2nd resonance, $\omega_1 = 2\omega$. AFC of the first harmonics are presented by the relationship $A_1 = A_1(\omega)$, where $A_1 = \sqrt{a_1^2 + b_1^2} = 2\sqrt{c_1 c_{-1}}$ (see Fig. 4). It is clear from the analysis of the AFC, that presence of the observed oscillations in the principal zone of instability can be considered as the result of pulling of stationary oscillations, self-excited in the zone of the second resonance. In other words, superposition between pulling zone and zone of the principal parametric resonance is the reason of the realization of this kind of oscillations.

VI. CONCLUSION

The analysis performed in this work confirms the existence in the Mathieu-Duffing system in the zone of the principal parametric resonance of two pairs opposite each other almost harmonic solutions with the period equal to the period of parametric excitation. Only the first couple of these solutions has a certain stability margin, solutions of the second pair are unstable. Despite the nearly harmonic nature of the regimes, their qualitative understanding and compliance with the numerical results can be achieved only with the allowance of higher harmonic components in the analytical solutions. It can be noted, that the revealed excitation mechanism, based on the phenomenon of oscillation pulling peculiar to nonlinear systems, makes such anomalous behavior to be an ordinary and a quite predictable one (at a small level of damping).

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Valērijs Belovodskis, Maksims Suhorukovs. Par Matjē-Dufinga vienādojuma jauniem periodiskiem risinājumiem galvenajā nestabilitātes zonā

Raksta mērķis – iepazīstināt ar pētījuma rezultātiem par Matjē-Dufinga vienādojuma jauno risinājumu eksistēšanas īpašībām parametriskās rezonanses galvenajā zonā, proti – par iespējām realizēt stacionārus risinājumus ar frekvenci, sakrītošu ar parametriskās ierosmes frekvenci. Rezultāti iegūti, kombinējot skaitlisko un analītisko pētījumu metodes, bet to pārbaude izpildīta, tieši integrējot sākotnējo vienādojumu. Sākotnēji tika izpildīta iegūto risinājumu monoharmoniskā analīze, kas deva pretrunīgus un savstarpēji nesaskaņotus rezultātus, un tikai turpmākā poliharmoniskā pieeja ļāva izdarīt viennozīmīgus secinājumus.

Валерий Беловодский, Максим Сухоруков. О новых периодических решениях уравнения Матъе-Дуффинга в главной зоне неустойчивости

Цель статьи — изложить результаты исследования вопросов существования некоторых новых решений уравнения Матъе-Дуффинга в зоне главного параметрического резонанса, а именно, — стационарных решений с частотой, равной частоте параметрического возбуждения. Они получены комбинированием численных и аналитических методов исследования, а их верификация производится путем непосредственного интегрирования исходного уравнения. Первоначально выполненный моногармонический анализ изучаемых решений дал противоречивые и не согласующиеся между собой результаты, и только последующий полигармонический подход дал возможность сделать однозначные заключения.