Determination of Heat Source Intensity inside a Plate

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***Abstract* - There is a method described in the article that allows calculating intensity of a heat source by using temperature measurements inside a body. A function of heat source intensity is defined as a Fourier series. Solution of the problem is obtained by means of well known solution of the nonhomogenous equation of heat conductivity. Coefficients of Fourier series are given by formula which contains only derivatives by time of measured data and calculated data not connected with experimentally obtained data.**

***Keywords* - thermal conductivity problem, heat source intensity, temperature measurements, Fourier series.**

I. Introduction

Origin of heat sources is quite various: electrical heating, chemical reactions, mechanic load etc. Determination of heat source intensity as per temperature measurements is one of inverse thermal conduction types. Such problem is discussed in monograph [1] and papers [2], [3]. These works discuss how a function of heat source intensity may be obtained assuming that temperature field and thermo physical characteristics are known.

II. Problem Definition

Temperature distribution in a plate containing a heat source can be described by the following equation:

$$c\rho\frac{\partial t}{\partial\tau} = \lambda\frac{\partial^2 t}{\partial x^2} + w(x,\tau), \tag{1}$$

where

$c\rho\left[\frac{J}{m^3\cdot k}\right]$ - thermal capacity of substance,

$\lambda\left[\frac{W}{m\cdot k}\right]$ - heat conductivity coefficient,

$t(x,\tau)$ – temperature,
$x\in[0,b]$ – coordinate,
τ – time,

$w(x,\tau)\left[\frac{W}{m^3}\right]$ - heat source intensity.

Hereafter it is presumed that temperature inside a body $t(x,\tau)$ is known. It can be calculated by making measurements by thermocouples and applying interpolation if necessary. In the same way let us assume that temperature on the border is known:

$$t(0,\tau)=t_0(\tau),\ t(b,\tau)=t_1(\tau) \tag{2}$$

and temperature is constant throughout sample at the beginning $t(0,\tau)=0$. One also assumes that thermo- physical characteristics of a material c,ρ,λ are known.

Problem is to find intensity of a heat source $w(x,\tau)$ knowing the above mentioned values. Problem can be easily formulated applying non-dimensional values:

$$\frac{\partial T}{\partial F} = \frac{\partial^2 T}{\partial N^2} + W(N,F) \tag{3}$$

$$T(0,F)=T_0(F),\ T(1,F)=T_1(F),\ T(N,0)=0, \quad 4)$$

where

$N=\frac{x}{b}$, $N\in[0,1]$ - non-dimensional coordinate,

$F=\frac{a\tau}{b^2}$ - non-dimensional time,

$T(N,F)=\frac{t(x,\tau)}{t_m}$ - non-dimensional temperature,

t_m – maximum measured temperature,

$W(N,F)=\frac{w(x,\tau)b^2}{\lambda\cdot t_m}$ - non-dimensional intensity of a heat source,

$T_0(F)=\frac{t_0(\tau)}{t_m}$, $T_1(F)=\frac{t_1(\tau)}{t_m}$ - non-dimensional temperature on the border.

Solution to problem (3), (4) consists of two addends [3]:

$$T(N,F)=T_b(N,F)+T_w(N,F), \tag{5}$$

where $T_b(N,F)$ is homogenous solution of the problem with non-homogenous boundary conditions:

$$\frac{\partial T_b}{\partial F} = \frac{\partial^2 T_b}{\partial N^2} \tag{6}$$

$$T_b(0,F)=T_0(F),\ T_b(1,F)=T_1(F),\ T_b(N,0)=0 \tag{7}$$

and $T_w(N,F)$ is non-homogenous equation:

$$\frac{\partial T_w}{\partial F} = \frac{\partial^2 T_w}{\partial N^2} + W(N,F) \quad (8)$$

Solution with the boundary conditions:

$$T_w(0,F) = T_w(1,F) = T_w(N,0) = 0\,. \quad (9)$$

The problem (6), (7) is solved [4] as follows:

$$T_b(N,F) = 2\pi \sum_{k=1}^{\infty} (-1)^{k+1} k \cdot$$

$$\cdot \int_0^F \left(T_0(\xi)\sin k\pi(1-N) + T_1(\xi)\sin k\pi N\right) e^{-k^2\pi^2(F-\xi)} d\xi \quad (10)$$

It follows from (10) that it depends only on measured boundary conditions, thus is calculable. Whereas $T_w(N,F)$ depends only on heat source intensity and does not depend on temperature on the borders:

$$T_w(N,F) = 2\sum_{k=1}^{\infty}\left(\int_0^F\left(e^{-k^2\pi^2(F-\varphi)}\int_0^1 W(\nu,\varphi)\sin k\pi\nu d\nu\right)d\varphi\right)\sin k\pi N \quad (11)$$

In this article a function of heat source intensity is expressed as a Fourier series:

$$W(N,F) = \sum_{k=1}^{\infty} b_k(F)\sin k\pi N \quad (12)$$

Amplitudes $b_k(F)$ obviously pertain to the function of heat source intensity as follows:

$$b_k(F) = 2\int_0^1 W(\nu,F)\sin k\pi\nu d\nu \quad (13)$$

To calculate heat source intensity, finding functions $b_k(F)$ is sufficient in extension of Fourier series (12). These functions must be expressed by measurable values.

III. Solution

It obviously follows from (5) that:

$$T_w(N,F) = T(N,F) - T_b(N,F). \quad (14)$$

It follows from (14), (13) and (11) that:

$$\sum_{k=1}^{\infty}\left(\int_0^F b_k(\varphi)e^{-k^2\pi^2(F-\varphi)}d\varphi\right)\sin k\pi N = T(N,F) - T_b(N,F). \quad (15)$$

Fourier series is located on the left side of the last equality, while data to be obtained in the experiment are located on the right side.

Both side of equality (15) are multiplying by $\sin n\pi N$ and integrated between *0* and *1*. Then, following orthogonality of trigonometric function, we obtain

$$\frac{1}{2}\int_0^F b_n(\varphi)e^{-n^2\pi^2(F-\varphi)}d\varphi = \int_0^1 T(N,F)\sin n\pi N dN -$$

$$-\int_0^1 T_b(N,F)\sin n\pi N dN \quad (16)$$

Defining

$$\int_0^1 T(N,F)\sin n\pi N dN = T_n(F) \quad (17)$$

and taking into account that

$$\int_0^1 T_b(N,F)\sin n\pi N dN = n\pi\int_0^F\left(T_0(\varphi) + (-1)^{n+1}T_1(\varphi)\right)e^{-n^2\pi^2(F-\varphi)}d\varphi$$

after simple modifications equality (16) can be rewritten as follows:

$$\int_0^F b_n(\varphi)e^{n^2\pi^2\varphi}d\varphi = 2T_n(F)e^{n^2\pi^2F} -$$

$$-n\pi\int_0^F\left(T_0(\varphi) + (-1)^{n+1}T_1(\varphi)\right)e^{n^2\pi^2\varphi}d\varphi$$

Having differentiated the last equality as per F and defined $b_n(F)$ wherefrom, one obtains:

$$b_n(F) = 2\left(n^2\pi^2T_n(F) + T_n'(F) - n\pi\left(T_0(F) + (-1)^{n+1}T_1(F)\right)\right).$$

IV. Conclusion

An expression of heat source intensity is determined using unstable temperature measurements inside a plate. This expression is formulated as a Fourier series by sinuses with unstable amplitudes. This expression contains only measureable values and the values calculable there from.

References

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I. Iltiņš, M. Iltiņa. Siltuma avota intensitātes noteikšana plāksnes iekšienē
Rakstā paradīta metode siltuma avota intensitātes noteikšanai, izmantojot temperatūras mērījumus viendimensiju plāksnes iekšienē. Šādus mērījumus var izdarīt ar termopāriem, kuru skaits, kā viegli saprotams, ir ierobežots. Tas apgrūtina siltuma avota intensitātes funkcijas noteikšanu tiešā veidā no siltumvadīšanas vienādojuma, jo nav iespējams atrast izmērītās temperatūras otro atvasinājumu pēc koordinātes ar pieņemamu precizitāti, jāatceras, ka termopāru skaits ir stipri ierobežots. Tātad atvasinājumu pēc koordinātes varētu noteikt tikai tajos punktos, kuros ir izvietoti termopāri. Mēs meklējām citu pieeju. Tāpēc šajā rakstā siltuma avota intensitātes funkcija tiek meklēta Furjē rindas pa sinusiem veidā. Šāda veida pieeja dod iespēju izvairīties no izmērītās temperatūras atvasināšanas pēc koordinātes, kas satur ļoti būtisku kļūdu. Furjē rindas koeficienti tiek iegūti formulas veidā, kura satur tikai izmērīto lielumu atvasinājumus pēc laika un aprēķināmus lielumus, kuri nav saistīti ar eksperimentāliem mērījumiem. Problēmas risinājums balstās uz nehomogēna siltumvadīšanas vienādojuma vispārzināmu atrisinājumu ar pirmā veida robežnosacījumiem, kas tiek piemērots dotajam uzdevumam. Tiek pieņemts, ka visi materiāla siltumfizikālie raksturlielumi, tas ir, siltumvadīšanas koeficients, temperatūrvadīsanas koeficients un siltumietilpība, ir zināmi. Temperatūra uz plāksnes robežām tiek izmērīta ar termopāriem.

И.Илтиньш, М. Илтиня. Определение интенсивности источника тепла внутри пластины
В статье рассмотрен метод определения интенсивности источника тепла по измерениям температуры внутри одномерной пластины. Такие измерения можно сделать при помощи термопар, число которых ограничено. Это затрудняет определение функции интенсивности источника тепла в прямом виде из уравнения теплопроводности, так как невозможно найти вторую производную измеренной температуры с приемлемой точностью. Нами предложен другой подход при решении данной проблемы. В этой статье функция интенсивности источника тепла определяется в виде ряда Фурье по синусам. Такой подход дает возможность избежать дифференцирования измеренной температуры по координате, что включает в себя существенную погрешность. Коэффициенты ряда Фурье определяются в виде формулы, которая содержит только производные измеренных температур по времени и вычисляемые величины, которые не связаны с экспериментальными измерениями. Решение задачи, построено основываясь на общеизвестном решении неоднородного уравнения теплопроводности с граничными условиями первого рода применительно к данной задаче. Предполагается, что все теплофизические характеристики материала, то есть коэффициент теплопроводности и теплоемкость, известны. Температуру на концах пластины измеряют при помощи термопар.