doi: 10.2478/v10313-012-0016-5

Wavelet Transform Modulus Maxima Approach for World Stock Index Multifractal Analysis

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***Abstract* –This paper describes an approach that is able to fix difference in multifractal behaviour of various World Stock Indexes. The approach is beneficial for the forecasting and simulations of the most European and Asian stock indexes. Multifractal analysis is provided using the so-called Wavelet Transform Modulus Maxima approach, which involves two basic aspects: Wavelet aspect (Direct Continuous Wavelet Transform, Skeleton construction) and Multifractal formalism (Fractal Partition Function calculation, Moment Generating Function calculation, Multifractal Spectrum estimation).**

***Keywords* – Wavelet Transform Modulus Maxima approach, Direct Continuous Wavelet Transform, Skeleton, Multifractal formalism, Fractal Partition Function, Moment Generating Function, Multifractal Spectrum, Stock indexes**

I. Introduction

Complex patterns and signals can be efficiently represented by decomposing them in different frequencies. The conventional method for this approach is the Fourier analysis. However, a certain instance may vary in time and space over the frequencies. In such cases, it is more appropriate to decompose the signal using another approach, which allows for spatial variation in the spectral composition of the signal. Examples of such approaches are the Windowed Fourier Approach and the multiresolution analysis approach, based on the so-called wavelets. Wavelet analysis provides important information about the mathematical morphology of a signal. An important method based on wavelet analysis is the Wavelet Transform Modulus Maxima (hereinafter WTMM). Using this method it is possible to describe the characteristic elements of a complex quasi-periodic signal. This description can then be used to recognize these elements in new signals.

Most of the information in a signal is carried by its irregular structures and its transient phenomena, called singularities. A method that excels in finding and identifying these singularities is the Wavelet Transform, because of its capability of decomposing a signal into elementary building blocks that are well localized in both time and frequency. Because of this capability, the Wavelet Transform is capable of defining the local regularity of a signal [1].

WTMM approach is a method for detecting the fractal dimension of a signal. More than this, the WTMM is capable of partitioning the time and scale domain of a signal into fractal dimension regions, and the method is sometimes referred to as a "mathematical microscope" due to its ability to inspect the multi-scale dimensional characteristics of a signal and possibly inform about the sources of these characteristics. The WTMM method uses continuous wavelet transform rather than Fourier transforms to detect singularities – that is discontinuities, areas in the signal that are not continuous at a particular derivative. This method is useful, when analyzing multifractal signals, that is, signals having multiple fractal dimensions [2].

WTMM consist of the following steps:

1) Data Mining and pre-processing, (stock index data should be minded, stock index prices should be represented in log-scale);

2) Wavelet analysis, (Direct Continuous Wavelet Transform procedure should be done, after that Skeleton should be constructed);

3) Multifractal Formalism, (Thermodynamic partition function estimation (Fractal Partition Function calculation); Scaling exponential function estimation (Moment Generating Function calculation); Multifractal spectrum estimation).

The rest of the paper is organized as follows: first of all, roots of stock index multifractality (the Multifractal Model of Asset Returns) are explored, after that WTMM approach pre-processing is illuminated, then both aspects of WTMM (Wavelet analysis and Multifractal Formalism) are narrated. After that WTMM approach is illustrated in respect to stock index multifractal analysis exemplified by German DAX30 stock index data. At the end of the paper, stock index multifractal analysis is performed.

II. Multifractal Model of Asset Returns

There is one general assumption about stock price behaviour. Expect the price of a stock or any other asset trading on the stock market is a multifractal process with fat tails and long-term dependency. Assumption about multifractal behaviour of stock indexes goes to the late 1990s and is based on Multifractal Model of Asset Return (MMAR), which was maintained by B. Mandelbrot, L. Calvet and A. Fisher. The Multifractal Model of Asset Returns (MMAR) provides the price of the asset by compounding a Fractional Brownian Model with a Trading Time. The Trading Time is a multifractal deformation of the time.

In accordance with MMAR stochastic process X(t) is called multifractal, if it has stationary increments and it satisfies:

$$E[(|s(t)|)^q] = c(q) \cdot t^{\tau(q)+1}, \qquad (1)$$

where: t – the time;
q – some non negative number, $q \in [0,1]$;
τ – the local scaling exponent;
s – the stochastic process, (signal);
c – the moment coefficient, independent of t. [3]

In order to satisfy equation and estimate the local scaling exponent of financial time series data τ Wavelet Transform Modulus Maxima (WTMM) approach is used. WTMM is a very advantageous approach for local scaling exponent estimation that allows building of the local scaling exponential function $\tau(q)$ for both: positive and negative q values.

III. Data Mining and Pre-Processing

For local scaling exponent estimation in case of financial time series, first of all, time series or the so-called signal should be represented in log-scale, e.g. in natural logarithmic scale in accordance with a formula:

$$s(t) = \ln(P_{close}(t)), \tag{2}$$

where: s(t) – the (pre-processed) signal;
t – the time at which the signal is recorded;
P_{close} – the market closing price.

There is one question under discussion, which price in the market bears more information in it. The authors consider closing prices to be suitable for local scaling exponent estimation.

IV. Wavelet Analysis

Direct Continuous Wavelet Transform (Direct CWT) procedure is implemented by a formula:

$$W(a,b) = \frac{1}{\sqrt{a}} \cdot \int_0^T \left(\psi\left(\frac{t-b}{a} \right) \cdot s(t) \right) dt \tag{3}$$

where: W – the wavelet coefficient(s);
a – the scaling parameter;
b – the shift parameter;
s – the signal;
t – the time at which the signal is recorded;
T – maximal time value;
ψ (a,b,t) – the mother wavelet (mother wavelet function).

A. Scaling Parameter

Scaling parameter a is representative for $a \in [1, T/2)$, but there is an opinion, that scaling parameter a used in WTMM approach is limited: $a \le 128$ [4]. The authors consider the most informative scaling parameters should be in interval $a \in [1, T/2]$, but in order to reduce calculation time, scaling parameters can be in interval $a \in [1, T/4]$. Consider that it is not advisable to calculate wavelet coefficients for large scales, which do not hold local maxima lines, because maximal scales are dependent on local maxima lines. However, local maxima lines are calculated using wavelet coefficients. To detect maximal scales, the authors propose "spy" local maxima lines for some scales.

The aim of "spy" algorithm is to detect number of local maxima lines at selective scales. Scale number is increased (using some step) until the number of local maxima lines at current scale reaches some minimal limit (default minimal number of local maxima lines, or percent of local maxima lines at the first scale (a = 1).

$$Nlcmx_{\min} = \sum_b LCMX_{a_{\max}, b}, \tag{4}$$

$$Nlcmx_{\min} = \sup(\{truncate(e^{(1-P)\cdot \ln(\sum_b LCMX_{1,b})}), N_{def}\}),$$

where: $Nlcmx_{min}$ – the minimal limit (minimal number) of local maxima lines;
N_{def} – the default minimal number of local maxima lines;
LcMx – the wavelet skeleton function;
a – the scaling parameter;
$a_{\max}$ – the maximal scaling parameter;
b – the shift parameter;
P – the percentage of local maxima lines considered.

In the current algorithm, a minimal limit number is found, using exponential interpolation. Consider, the percentage of used local maxima lines at the first scale (a = 1) is equal to 0, but the percentage of used local maxima lines at the last scale (first such scale, when a number of local maxima lines reaches 1) is equal to 100; consequently, minimal limit number, at certain percent of used local maxima lines, is found using exponential interpolation (by increasing scale, the number of local maxima lines decreases exponentially).

B. Shift Parameter b and Mother Wavelet Function ψ

Shift parameter b is limited $b \le T$, because shifting parameter b cannot be greater than time T, at which the signal is recorded. For detailed exploration of Direct CWT with numerical examples in MathCad, see resource [5].

Typical mother wavelet functions used for WTMM approach are MHAT (Mexican Hat) and Wave (first order Gauss wavelet).

Wave mother wavelet function is considered in a formula:

$$\psi(t) = -t \cdot e^{-\frac{t^2}{2}}, \tag{5}$$

where: ψ – the Wave mother wavelet function;
t – the time at which the signal is recorded.[6]

MHAT mother wavelet function is as follows:

$$\psi(t) = (1 - t^2) \cdot e^{-\frac{t^2}{2}}, \tag{6}$$

where: ψ – the Wave mother wavelet function;
t – the time at which the signal is recorded.[7]

There are many other mother wavelet functions, which are not described here because of complexity of exploration. Many authors consider better results of WTMM approach to be afforded using Wave mother wavelet function [8, 9]. The authors consider Daubechies and Morlet mother wavelet function also to be appropriate for WTMM approach.

Direct CWT provides wavelet coefficients *W(a,b)*, which can be written in a matrix form:

$$W_{a,b} = W(a,b) \mid (a,b \in N) \wedge (a \in [1, a_{\max}]) \wedge (b \in [1,T]), \quad (7)$$

where: W(a,b) – the wavelet coefficients;
a – the scaling parameter;
$a_{\max}$ – the maximal scaling parameter;
b – the shift parameter;
T – the signal length.

Also the absolute wavelet coefficient matrix is used, which can be calculated using a formula:

$$W^{sq}{}_{a,b} = (W(a,b))^2 \mid (a,b \in N) \wedge (a \in [1, a_{\max}]) \wedge (b \in [1,T]). \quad (8)$$

Squared wavelet coefficient matrix also has sense in the context of wavelet skeleton. This matrix is calculated by a formula:

$$W^{sq}{}_{a,b} = (W(a,b))^2 \mid (a,b \in N) \wedge (a \in [1, a_{\max}]) \wedge (b \in [1,T]), \quad (9)$$

where: W^{sq} – the squared wavelet coefficients matrix;
W(a,b) – the wavelet coefficients;
a – the scaling parameter;
$a_{\max}$ – the maximal scaling parameter;
b – the shift parameter;
T – the signal length.

Wavelet coefficient (usually absolute wavelet coefficient) matrix is displayed as 3D graph projection on $a \otimes b$ space. Also, in generated output plot wavelet coefficients are coloured by their absolute values.

C. Skeleton Construction

Wavelet Skeleton is an aggregate of all Local Maxima Lines (LML) on each scale of Wavelet coefficient matrix. The idea of Skeleton matrix construction is to remove all wavelet coefficients in absolute wavelet coefficients matrix that are not maximal. In this way, in Skeleton matrix there are only those absolute wavelet coefficients that belong to local maxima lines.

Next skeleton function *(LcMx)* is considered:

$$LcMx_{a,b} = \begin{cases} 1 \mid \dfrac{\partial (W(a,b))^2}{\partial b} = 0 \\ 0 \mid \neg(\dfrac{\partial (W(a,b))^2}{\partial b} = 0) \end{cases}, \quad (10)$$

Under the conditions:

$$(a,b \in N) \wedge (a \in [1, a_{\max}]) \wedge (b \in [1,T]$$

where: LcMx – the wavelet skeleton function;
W(a,b) – the wavelet coefficients;
a – the scaling parameter;
$a_{\max}$ – the maximal scaling parameter;
b – the shift parameter;
T – the signal length.[10]

Wavelet Skeleton function can be written in a matrix form by an equation:

$$LcMx_{a,b} = LcMx(a,b) \mid \ldots$$
$$\ldots(a,b \in N) \wedge (a \in [1, a_{\max}]) \wedge (b \in [1,T]), \quad (11)$$

where: LcMx – the wavelet skeleton function;
a – the scaling parameter;
$a_{\max}$ – the maximal scaling parameter;
b – the shift parameter;
T – the signal length.

Usually Skeleton function in a matrix form is calculated from squared wavelet coefficients matrix by a formula:

$$LcMx_{a,b} = \begin{cases} 1 \mid (W^{sq}{}_{a,b+1} - W^{sq}{}_{a,b}) < \varepsilon \\ 0 \mid (W^{sq}{}_{a,b+1} - W^{sq}{}_{a,b}) > \varepsilon \end{cases}, \quad (12)$$

Under the conditions:

$$(a,b \in N) \wedge (a \in [1, a_{\max}]) \wedge (b \in [1, T-1]$$

where: LcMx – the wavelet skeleton function;
a – the scaling parameter;
$a_{\max}$ – the maximal scaling parameter;
b – the shift parameter;
T – the signal length.

Skeleton matrix is a scope of all local maxima points that exist on each scale a. In fact, skeleton function is a logical function, that has only two variables {0,1}. One is used if Skeleton matrix element is local maximum, zero – otherwise. Skeleton matrix 3D graph projection on $a \otimes b$ space resembles the map, where Local Maxima Lines (LML) are notched.

For skeleton function construction in Matlab environment, build-in function localmax is used, which is improved in Matlab code. Full code is available in the article [11].

LML Matrix "corners" $LcMx_{a,b} \mid (T - a \le b) \vee (b \le a)$ need to be cleared (ignored) in skeleton. Consider, wavelet coefficients on corners provide little information; consequently, local maxima lines on corners should be removed, see formula:

$$LcMx_{a,b} = \begin{cases} 1 \mid (\dfrac{\partial (W(a,b))^2}{\partial b} = 0) \wedge (a < b < T - a) \\ 0 \mid \neg(\dfrac{\partial (W(a,b))^2}{\partial b} = 0) \vee (T - a \le b) \vee (b \le a) \end{cases}, \quad (13)$$

Under the conditions:

$$(a,b \in N) \wedge (a \in [1,a_{\max}]) \wedge (b \in [1,T]$$

where: LcMx – the wavelet skeleton function;
a – the scaling parameter;
$a_{\max}$ – the maximal scaling parameter;
b – the shift parameter;
T – the signal length.

Skeleton matrix points generate ‘broken LML’. In order to obtain quick results for “dirty” wavelet modulus maxima coefficients, use the following formula:

$$WTMM^{fast}{}_{a,b} = W^{abs}{}_{a,b} \cdot LcMx_{a,b}, \tag{14}$$

Under the conditions:

$$(a,b \in N) \wedge (a \in [1,a_{\max}]) \wedge (b \in [1,T-1]$$

where: WTMM^{fast} – the “dirty” wavelet modulus maxima coefficients (fast algorithm);
LcMx – the wavelet skeleton function;
W^{abs} – the absolute wavelet coefficients matrix;
a – the scaling parameter;
$a_{\max}$ – the maximal scaling parameter;
b – the shift parameter;
T – the signal length.

Fast algorithm of WTMM approach has one disadvantage, LML is broken that impacts Fractal Partition Function construction.

The latest methods in WTMM approach expect that Wavelet coefficients grow consequently on LML: by increasing the scaling parameter a $\forall b, a1 < a2 < a_{\max}$ almost everywhere wavelet coefficients increase $|W(a1,b)| < |W(a2,b)| < |W(a_{\max})|$. However, in some points of broken LML, *LcMx* = 0, consequently $WTMM^{fast}$- = 0.

In order to prevent this case, in WTMM approach the following procedures are used:

1) remove all “gaps” in *LcMx* matrix;
2) trace whether wavelet modulus coefficients on LML grow consequently with scaling parameter a: $\forall b, a1 < a2 < a_{\max}$ almost everywhere $|W(a1,b)| < |W(a2,b)| < |W(a_{\max})|$.

In fact, the last two procedures are the darkest sides of WTMM approach in literature. There is supremum formula for fractal partition function calculation.

$$Z_{q,a} = \sum_{l \in L(a)} (\sup_{\substack{b,a' \in l \\ a' \le a}} \|W(a',b)\|)^q, \tag{15}$$

Under the conditions:

$$(a,b \in N) \wedge (a \in [1,a_{\max}]) \wedge (b \in [1,T-1]$$

where: Z – the fractal partition function;
W(a,b) – the wavelet coefficients;
a – the scaling parameter;
$a_{\max}$ – the maximal scaling parameter;
b – the shift parameter;
T – the signal length;
l – the local maxima line (LML);
L(a) – the scope of all local maxima lines that exist on scale a;
q – the power indicator – a certain number, e.g. $q \in [-5,5]$. See [12].

This formula suggests that for fractal partition function calculation those absolute wavelet coefficients should be taken, which:

1) lie on LML (LML l exists on $a' \in [1,a]$), and
2) are maximum, which exist on LML (LML exist on $a' \in [1,a]$)
3) exist on decent LML (LML exists on $a' \in [1,a]$), if a decent absolute wavelet coefficient is not maximal on this LML.

In general, this formula is not described analytically in literature. Most programming codes used for exploration of WTMM do not use “supremum algorithm” at all. Most of them use fast algorithm with “dirty” wavelet modulus maxima coefficients [13].

Supremum algorithms in practice are very sophisticated, because skeleton does not contain local maxima lines in the true sense of the word: it has disconnected broken lines and single points. Actually it is difficult to assume how one skeleton point or broken line is related to another broken line or point. This is a very hard programming task, which is arduous for narration in mathematical language. Instead, Matlab code is represented [11].

When “dirty” wavelet modulus maxima coefficients are found, one more additional procedure should be established – probability normalization condition. This condition is directly concerned with Multifractal formalism procedure. Absolute wavelet coefficient normalization formulas are the following:

$$\exists C(a) : \sum_{b=1}^{T-1} (C(a) \cdot WTMM_{a,b}) = 1, \tag{16}$$

$$C(a) \cdot \sum_{b=1}^{T-1} WTMM_{a,b} = 1, \tag{17}$$

$$C(a) = \frac{1}{\sum_{b=1}^{T-1} WTMM_{a,b}} = (\sum_{b=1}^{T-1} WTMM_{a,b})^{-1}, \tag{18}$$

Under the conditions

$$(a,b \in N) \wedge (a \in [1,a_{\max}]) \wedge (b \in [1,T-1]$$

where: WTMM – the wavelet modulus maxima coefficients;
C(a) – the constant depending on scaling parameter a;
a – the scaling parameter;
$a_{\max}$ – the maximal scaling parameter;
b – the shift parameter;
T – the signal length.

Absolute wavelet coefficient normalization procedure usually is not implemented in WTMM approach, but the authors consider it should be implemented in order to provide analyzable results from the point of view of Multifractal formalism.

Next, the supremum algorithm should be preceded. It consists of seven main steps:

1) Define matches (relations between single local maxima points)
2) Define match conflicting (one cell to more) and non-conflicting cases;
3) Create chains from pairs;
4) Chain interpolation (to fill missing wavelet coefficients on WTMM line);
5) Add points to LCMX map (on line gaps);
6) Add single points to LCMX map;
7) Change variables (in the end).

Supremum algorithm (code) is represented in the scientific journal [14], but here only main steps are indicated.

After this step, Multifractal formalism (MfF) should be applied to WTMM approach implementation.

V. Multifractal Formalism Part

Multifractal Formalism (MfF) procedure is part of WTMM approach and consists of the following steps:

1. Thermodynamic partition function estimation (Fractal Partition Function calculation);
2. Scaling exponential function estimation (Moment Generating Function calculation);
3. Multifractal spectrum estimation.

Here and further the MfF algorithm is considered:

D. Thermodynamic Partition Function Estimation

Thermodynamic partition function is based on ‘generalized’ partition function consideration. Wavelet modulus maxima coefficients are akin to probability measure in the so-called ‘generalized’ partition function.

Thermodynamic partition function is the following:

$$Z(q,a) = \sum_{b=1}^{T-1} (C(a) \cdot WTMM_{a,b})^q \mid \neg(WTMM_{a,b} = 0), \quad (19)$$

where: Z (q,a) – the thermodynamic partition function
WTMM – the wavelet modulus maxima coefficients;
C(a) – the constant depending on scaling parameter a;
a – the scaling parameter;
q – the exponential number, for example, $q \in [-5,5]$.

Thermodynamic partition function is a function of two arguments - scaling parameter a and power argument q. Power argument q is the set of zero-mean numbers (exponential number, for example in interval $q \in [-5,5]$) that indicate presence of wavelet modulus maxima coefficients of different values: the presence of relatively small wavelet modulus maxima coefficients $WTMM \to 0$ is detectable with large Z values in case of negative q values $q < 0$; and the presence of relatively large wavelet modulus maxima coefficients is detectable with large Z values $Z \to \infty$ in case of positive q values $q > 0$.

Scaling parameter a is an argument of thermodynamic partition function that, by its nature, is the scaling “etalon” for thermodynamic partition function value. Interdependence between thermodynamic partition function value and scaling parameter a discovers the scalability of signal. That is a key moment in the WTMM approach.

Thermodynamic partition function is finite if wavelet modulus maxima coefficients are not equal to zero: $\neg(WTMM_{a,b} = 0)$. In order to satisfy condition all zero coefficients should be neglected in wavelet modulus maxima matrix WTMM. The origin of zero coefficients in wavelet modulus maxima matrix is in LcMx wavelet skeleton function – all elements in Skeleton matrix that are not local maxima are zero valued elements. Taking into account the previous statement, thermodynamic partition function can be written as follows:

$$Z(q,a) = \sum_{b=1}^{T-1} (C(a) \cdot WTMM_{a,b})^q \mid (LcMx_{a,b} = 1), \quad (20)$$

where: Z(q,a) – the thermodynamic partition function
$WTMM$ – the wavelet modulus maxima coefficients;
LcMx – the wavelet skeleton function;
C(a) – the constant depending on scaling parameter;
a – the scaling parameter;
q – the exponential number, for example $q \in [-5,5]$.

In order to discover the scalability of signal, scaling exponential function will be considered in the next section.

E. Scaling Exponential Function Estimation

Scaling exponential function is one argument function, that indicate interdependence between thermodynamic partition function Z and scaling parameter a. Scaling exponential function is calculated as the slope between thermodynamic partition function Z and scaling parameter a by formula:

$$\tau(q) = \lim_{a \to 0} \frac{\ln(Z(q,a)}{\ln(a)}, \quad (21)$$

where: Z(q,a) – the thermodynamic partition function
τ – the local scaling exponent;
a – the scaling parameter;
q – the exponential number, e.g. $q \in [-5,5]$. See [15]

The previous formula can be expanded to:

$$\tau(q) = \frac{\sum\limits_{a_{\max}} (x(a) \cdot y(q,a))}{\sum\limits_{a_{\max}} (x(a))^2}, \tag{22}$$

where:

$$y(q,a) = \ln(Z(q,a)) - \frac{\sum\limits_{a=1} \ln(Z(q,a)}{a_{\max}}, \tag{23}$$

$$x(a) = \ln(a) - \frac{\sum\limits_{a=1} \ln(a)}{a_{\max}}, \tag{24}$$

under the conditions:

$$(a \in N) \wedge (a \in [1, a_{\max}])$$

see: τ – the local scaling exponent;
Z (q,a) – the thermodynamic partition function
a – the scaling parameter;
$a_{\max}$ – the maximal scaling parameter;
q – the exponential number, e.g. $q \in [-5,5]$.

Scaling exponential function is non-decreasing function that indicates the scalability of signal. There are two main cases: mono- and multifractal. In monofractal case, the scaling exponential function is linear. In multifractal case, the scaling exponential function is everywhere convex.

There is one interesting property to be considered with Scaling exponential function:

$$\tau(0) + 1 = 0 \tag{25}$$

where: τ – the local scaling exponent.

This condition is essential for Multifractal spectrum estimation. In fact, this condition is satisfied not in all cases, for example, in case of financial time series analysis. For this reason, the authors offer the following formula:

$$\exists k, \tau(0) + 1 = 0,$$
$$\tau(0) = \frac{\sum\limits_{a_{\max}} \left(\log_k(a) - \frac{\sum\limits_{a_{\max}} \log_k(a)}{a_{\max}} \cdot \ln(Z(0,a)) - \frac{\sum\limits_{a_{\max}} \ln(Z(0,a)}{a_{\max}}\right)}{\sum\limits_{a_{\max}} \left(\log_k(a) - \frac{\sum\limits_{a_{\max}} \log_k(a)}{a_{\max}}\right)^2} = -1; \tag{26}$$

Let:

$$k = -\tau(0)$$

then:

$$\tau(q) = \frac{\sum\limits_{a_{\max}} \left(\log_k(a) - \frac{\sum\limits_{a_{\max}} \log_k(a)}{a_{\max}} \cdot \ln(Z(q,a)) - \frac{\sum\limits_{a_{\max}} \ln(Z(q,a)}{a_{\max}}\right)}{\sum\limits_{a_{\max}} \left(\log_k(a) - \frac{\sum\limits_{a_{\max}} \log_k(a)}{a_{\max}}\right)^2}; \tag{27}$$

Or:

$$\tau(q) = \frac{\sum\limits_{a_{\max}} (x(a) \cdot y(q,a))}{\sum\limits_{a_{\max}} (x(a))^2}, \tag{28}$$

where:

$$y(q,a) = \ln(Z(q,a)) - \frac{\sum\limits_{a_{\max}} \ln(Z(q,a)}{a_{\max}};$$
$$x(a) = \ln(a^k) - \frac{\sum\limits_{a_{\max}} \ln(a^k)}{a_{\max}}; \tag{29}$$

under the conditions:

$$(a \in N) \wedge (a \in [1, a_{\max}])$$

see: τ – the local scaling exponent;
Z(q,a) – the thermodynamic partition function
a – the scaling parameter;
$a_{\max}$ – the maximal scaling parameter;
q – the exponential number, e.g. $q \in [-5,5]$.

This transformation is beneficial for scaling exponential function estimation in case if $\neg(\tau(0) + 1 = 0)$. After transformation, this essential condition is satisfied.

Estimated scaling exponential function is used for MMAR stochastic process simulation that satisfies equation (1).

F. Multifractal Spectrum Function Estimation

Multifractal formalism uses multifractal spectrum for the detailed fractal analysis of the signal. Multifractal spectrum function shows the scope of all fractal measures. Multifractal spectrum function is calculated from scaling exponential function via Legendre transformation by formula:

$$h = h(q) = \frac{\partial \tau(q)}{\partial q},$$
$$D(h) = \inf_q (q \cdot h - \tau(q)), \tag{30}$$

where: D – the multifractal spectrum;
h – the Holder exponent;
τ – the local scaling exponent;
q – the exponential number, e.g. $q \in [-5,5]$. [16]

In case of monofractal signal, the multifractal spectrum function transforms to a single point *(h, 1)*, but in case of multifractal signal, the multifractal spectrum function is bell-shaped function, whose branches are directed downwards. Multifractal spectrum function maximum point is *(h, 1)*. In both cases: mono- and multifractal h is Holder exponent, which indicates the most typical measure of fractal.

Multifractal spectrum can be approximated by a polynomial. Multifractal spectrum is approximated with 4th degree polynomial:

$$D(h) = \sum_{i=0}^{4} k_i \cdot h^i, \qquad (31)$$

where: D – the multifractal spectrum;
h – the Holder exponent;
k – the polynomial coefficients;
i – the polynomial coefficient index i = 0.1,…4 [17].

For all algorithms mentioned here, Matlab codes are provided, see [18, 19].

VI. Stock Index Multifractal Analysis

Here and further, the stock index multifractal analysis is performed using WTMM.

For current research, decent stock market closing prices have been taken. It is assumed that closing prices are the most representative, because most of deals are done by institutional investors at the end of trading day. However, there are also opinions expressed in some studies that intraday prices are more representative for time series research. Most of indexes are considered for the whole available period of time; see the table data (the Dow Jones Industrial Index has been taken into account only since 1980 because of large computations.)

TABLE I
Data on Research Objects

Index name	Code	Data available		Refe-rence
		from	to	
IBEX35 Index	IBEX35	1993.VII	2012.VI	[20]
DAX30 Index	DAX30	1990.XI	2012.VI	[21]
Swiss Market Index	SMI	1990.XI	2012.VI	[22]
CAC40 Index	CAC40	1990.III	2012.VI	[23]
FTSE100 Index	FTSE	1984.IV	2012.VI	[24]
Dow Jones Industrial Index	DJIA	1980.I	2012.VI	[25]
Amsterdam Exchange Index	AEX	1992.X	2012.VI	[26]
Hang Seng Index	HSI	1986.XII	2012.VI	[27]
NIKKEI225 Index	Nikkei225	1984.I	2012.VI	[28]
Straits Times Index {Singapore}	STI	1988.I	2012.VI	[29]
Philippine Stock Exchange Index	PSEI	2000.I	2012.VI	[30]
BSE India Sensex 30 Index	BSESN	1997.VII	2012.VI	[31]

Here and further, the stock index multifractal analysis algorithm is exemplified by DAX30 Index. The first index data are represented in log-scale, see Fig. 1.

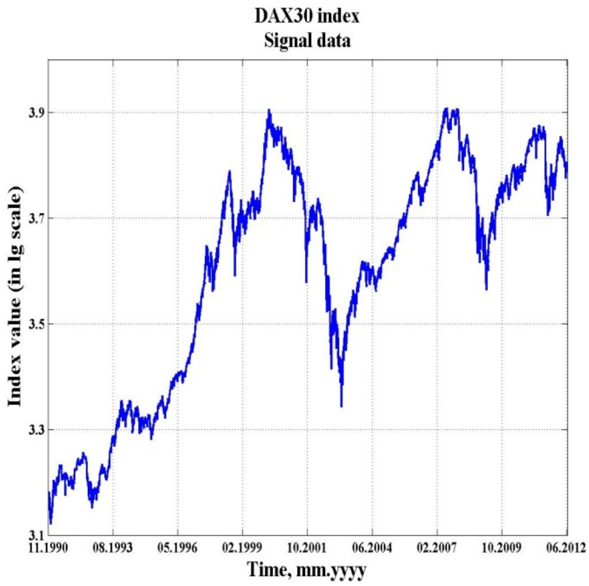

Fig. 1. DAX30 index data (represented in log-scale)

Direct Continuous Wavelet transform is illustrated in Fig. 2.

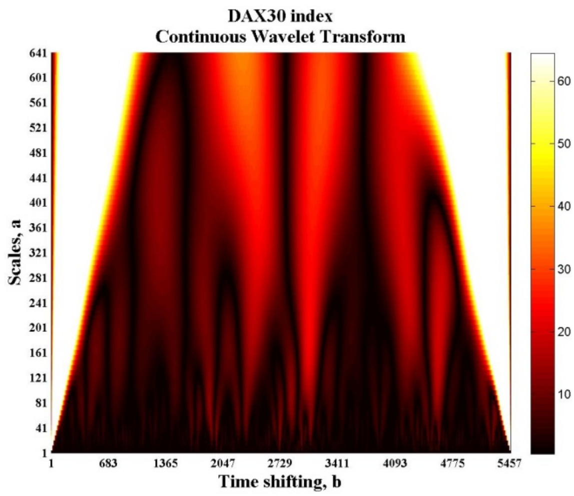

Fig. 2. DAX30 index continuous wavelet transform

DAX30 index continuous wavelet transform shows wavelet coefficients in decent shifting and scale parameters. Wavelet coefficients matrix can be represented in a three-dimensional graph, but it can also be represented as wavelet coefficient projection onto the plane formed by the shift and scaling parameters as it is implemented here. Wavelet coefficients are shown in their absolute values and coloured in accordance with colour bar. Dark colours correspond to lower absolute wavelet coefficient values. Light colours indicate higher absolute wavelet coefficient values. Wavelet coefficient matrix allows LML selection or Skeleton function construction, see Fig. 3.

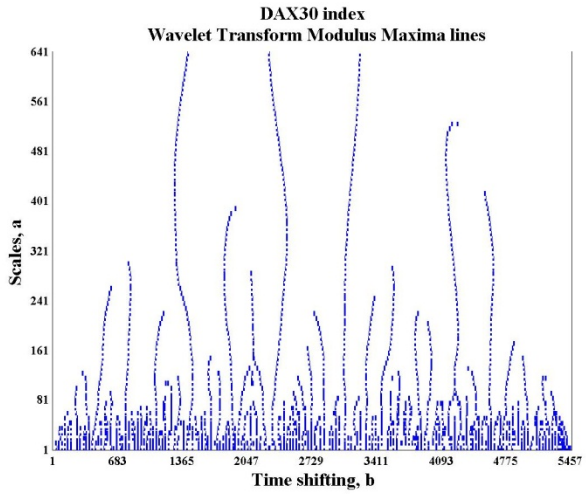

Fig. 3. DAX30 index skeleton function

Local maxima lines are constructed using Wavelet coefficient matrix, selecting local maxima points on each scale parameter. The scope of all local maxima lines builds the so-called Skeleton function. This function illuminates periodicity of the signal on decent scales. Naturally, stock indexes show periodicity affected by population cycles, economic cycles, moon cycles, stock exchange cycles and events. In general, the skeleton function shows the scalability of the signal. Local maxima lines are designed to select the skeleton (basics of wavelet coefficient matrix) in order to simplify multifractal analysis in the whole wavelet coefficient matrix to multifractal analysis within wavelet coefficients on the skeleton.

Since DAX30 index skeleton is constructed, thermodynamic partition function is estimated (see Fig. 4).

DAX30 index thermodynamic partition function is a three-dimensional graph – function of two arguments – scaling parameter a and power argument *q*. Power argument *q* is the set of zero-mean numbers (in interval $q \in [-5,5]$). Thermodynamic partition function is designed to eliminate the presence of wavelet modulus maxima coefficients of different values. The presence of relatively small wavelet modulus maxima coefficients $WTMM \to 0$ are detectable with large Z values $Z \to \infty$ in case of negative *q* values $q < 0$; and the presence of relatively large wavelet modulus maxima coefficients are detectable with large Z values $Z \to \infty$ in case of positive *q* values $q > 0$.

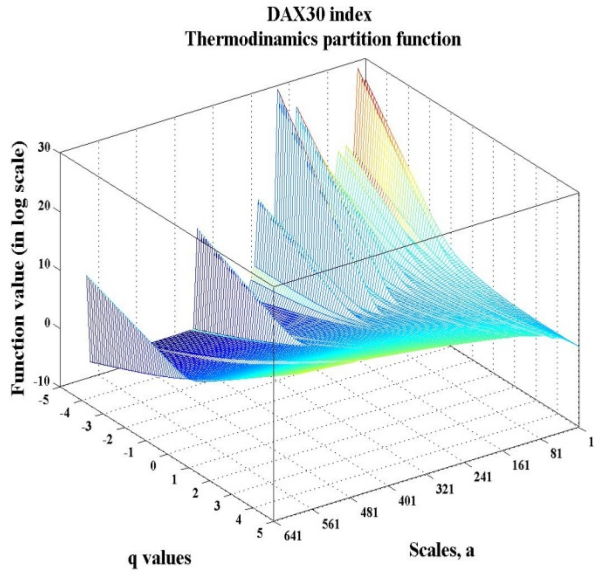

Fig. 4. DAX30 index thermodynamic partition function

Interdependence between thermodynamic partition function value and scaling parameter *a* discovers the scalability of signal. In order to discover the scalability of DAX30 index, local scaling exponential function is represented in Fig. 5.

DAX30 index scaling exponential function is everywhere convex that indicates multifractal behaviour of the index. Multifractal behaviour of stock index assumes that the index does not have some decent fractal measure, but is characterized by the scope of fractal measures. In case of monofractal behaviour, the scaling exponential function is line.

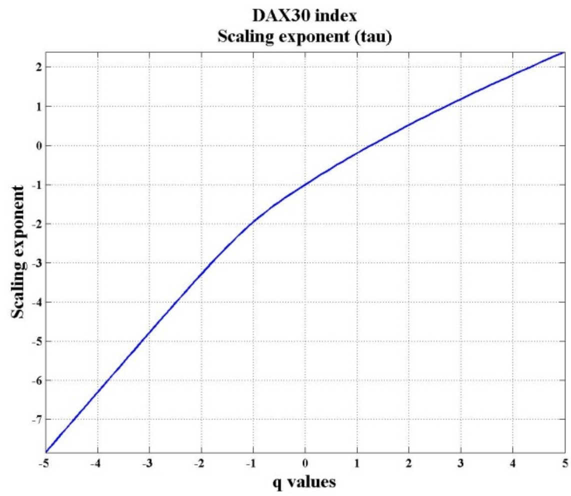

Fig. 5. DAX30 index local scaling exponential function

For detailed DAX30 index multifractal analysis, its multifractal spectrum is presented in Fig. 6.

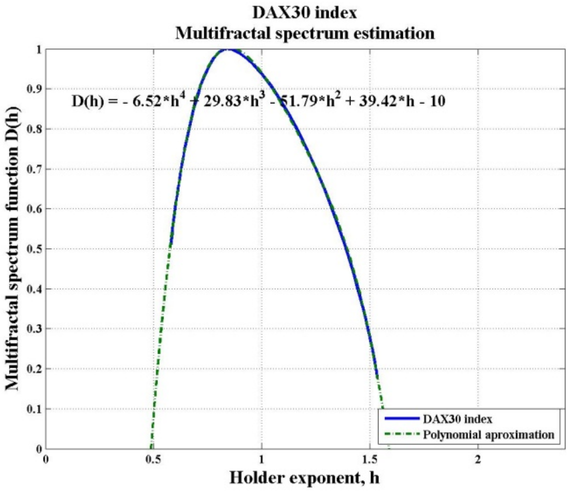

Fig. 6. DAX30 index multifractal spectrum estimation (polynomial approximation)

DAX30 index multifractal spectrum is bell-shaped function, whose branches are directed downwards, with high correlation; multifractal spectrum function is approximated with a 4th degree polynomial.

All steps of WTMM approach are performed for all objects of experiment. Multifractal spectra are approximated with a 4th degree polynomial, using (31).

Estimated multifractal spectra are analyzed in cross-correlation matrix using Pearson correlation coefficients.

VII. RESULTS

1. According to the multifractal analysis, all research objects demonstrate strong multifractal behaviour. Polynomial coefficients of multifractal spectrum approximations are shown in Table II.

2. Most typical fractal measures are discovered by h_0 value. This value shows the typical fractal measure of signal – higher h_0 values, which indicate anti-persistent properties, but lower values indicate the presence of fractal properties of time series, consequently, indicate persistent or trend sustained properties of time series. Spectrum maxima (h_0 indicator) for all indexes are about 0.77, $0.63 \leq h_0 \leq 0.89$ that indicate the presence of fractal properties of time series consequently that shows persistent or trend sustainable properties of indexes.

3. Fuzziness of the multifractal spectrum implies how regular a fractal is; if fuzziness of multifractal spectrum is low, fractal is quite regular. Fuzziness of the multifractal spectrum is indicated with Δh indicator; higher Δh indicators $(\Delta h > 1)$ are demonstrated by IBEX35 index, Hang Seng index, Straits Times Index {Singapore}. Lower Δh indicators are demonstrated by FTSE100 index and NIKKEI225 index. FTSE100 index and NIKKEI225 have more regular fractal properties, but IBEX35 index, Hang Seng index, Straits Times Index {Singapore} contain a wide scope of fractal measures.

4. Fuzziness of the multifractal spectrum is indicated also with higher k_4 polynomial coefficient. Higher k_4 polynomial coefficients $k_4 < -2.5$ are demonstrated by Straits Times Index {Singapore}, Philippines Stock Exchange Index, IBEX35 index and Hang Seng index. In fact, it is difficult to make any conclusion about fuzziness of the multifractal spectrum by k_4 polynomial coefficient because correlation between multifractal spectrum and its approximation should be taken into account. In lower approximation correlation case, Δh indicator should be helpful. However, the presence of both higher polynomial coefficient k_4 and higher Δh indicator clearly proves the fuzziness of multifractal spectrum, e.g. for Straits Times Index {Singapore}, IBEX35 index and Hang Seng index.

TABLE II
MULTIFRACTAL SPECTRUM APPROXIMATION WITH QUADRATIC FUNCTION, COEFFICIENTS AND SPECTRUM MAXIMA

Index code	Polynomial coefficients				
	k_4	k_3	k_2	k_1	k_0
IBEX35	-2.17	10.83	-20.89	17.48	-4.29
DAX30	-6.52	29.83	-51.79	39.42	-10.00
SMI	-5.42	24.17	-42.42	33.25	-8.64
CAC40	-6.21	28.52	-49.70	38.19	-9.83
FTSE100	-16.52	67.11	-103.02	69.49	-16.26
DJIA	-7.75	32.89	-52.18	35.65	-7.80
AEX	-2.68	11.33	-19.53	15.12	-3.31
HSI	-2.54	10.59	-17.52	12.78	-2.40
NIKKEI225	-16.41	62.61	-88.39	53.69	-10.76
STI	-2.10	8.03	-12.65	9.00	-1.37
PSEI	-2.08	10.78	-21.93	17.45	-3.70
BSESN	0.34	1.14	-7.72	8.31	-1.53

TABLE III
MULTIFRACTAL SPECTRUM FUZINESS AND MAXIMA

Index code	Holder exponent			
	h_0	h_{min}	h_{max}	Δh
IBEX35	0.89	0.46	1.80	1.34
DAX30	0.85	0.58	1.53	0.95
SMI	0.88	0.57	1.46	0.89
CAC40	0.89	0.61	1.60	0.99
FTSE100	0.74	0.55	1.32	0.77
DJIA	0.70	0.49	1.49	1.01
AEX	0.85	0.44	1.52	1.08
HSI	0.80	0.34	1.60	1.26
NIKKEI225	0.59	0.50	1.35	0.85
STI	0.75	0.25	1.52	1.27
PSEI	0.64	0.39	1.15	0.76
BSESN	0.63	0.27	1.10	0.83

Analogically, the presence of both lower polynomial coefficient k_4 and lower Δh at the same time indicate regularity of fractal properties, e.g. for FTSE100 index and NIKKEI225.

Cross-correlation research results are shown in Table IV.

TABLE IV

MULTIFRACTAL SPECTRA CROSS-CORRELATION MATRIX (IN PERCENTAGE)

	IBEX35	DAX30	SMI	CAC40	FTSE100	DJIA	AEX	HIS	NIKKEI225
DAX30	99.8								
SMI	99.6	99.0							
CAC40	99.3	99.8	98.1						
FTSE100	96.2	95.6	98.1	93.5					
DJIA	95.2	94.7	97.0	92.3	99.5				
AEX	91.3	89.3	94.4	86.2	97.8	97.2			
HSI	83.7	81.4	88.0	77.3	94.2	94.5	98.6		
NIKKEI	79.3	77.7	84.3	73.5	92.8	92.8	95.7	97.9	
STI	63.6	60.2	70.0	54.9	80.1	80.4	89.6	95.2	94.6
PSEI	42.4	39.1	49.2	32.7	62.3	65.3	74.1	84.2	83.2
BSESN	17.6	13.9	25.1	7.1	39.7	43.3	54.5	67.5	66.6

According to multifractal spectrum cross-correlation matrix, almost all research objects prove very strong correlation in multifractal spectra (Asian ex small indexes: Straits Times Index {Singapore}, PSEI Index {Philipines}, BSE India Sensex 30 Index). All indexes can be roughly divided into three main groups: European indexes (DAX30 Index, Swiss Market Index, CAC40 Index, FTSE100 Index), Global group including European indexes and Dow Jones Industrial Index, Amsterdam Exchange Index, Hang Seng Index, NIKKEI225 Index, and Asian indexes (BSE India Sensex 30 Index, PSEI Index {Philipines}, Straits Times Index {Singapore}, Hang Seng Index, NIKKEI225 Index). Some of these indexes are included in various groups.

Although all indexes are represented for various periods of time, multifractal spectrum cross-correlation matrix can classify stock indexes by their fractal properties. Suppose, related indexes are very similar in their behaviour: that assumes an 'ability' to fix, hold and maintain market information in a 'certain way'. Practically, it means that all indexes are interdependent, global and operate as a single organism.

VIII. ACKNOWLEDGEMENTS

This work is partly funded by the ERDF (ERAF) project No.2011/0008/2DP/2.1.1.1.0/10/APIA/VIAA/0 18 "Development of Insurance Distributed Software Based on Intelligent Agents, Modelling, and Web Technologies".

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Andrejs Pučkovs, Andrejs Matvejevs. **Veivlet pārveidojumu moduļu maksimumu pieeja pasaules akciju indeksu multifraktāļu analīzei**
Šī raksta mērķis ir nodrošināt pieeju, kas spēj izpētīt atšķirības pasaules akciju indeksu multifraktāļu spektros. Šī pieeja ir spējīga noteikt atšķirības multifraktāļu uzvedībā dažādiem pasaules akciju indeksiem. Šī pieeja ir piemērota Eiropas un Āzijas akciju indeksu prognozēšanai un imitācijas modelēšanai. Multifraktālā analīze ir īstenota, izmantojot tā saucamo veivlet pārveidojumu moduļu maksimumu pieeju, kas ietver sevī divus galvenos aspektus: veivlet analīzi un multifraktāļu algoritmu. Veivlet pārveidojumu moduļu maksimumu pieeja ir metode, kas atklāj signāla fraktāļu mēru. Šī raksta secība ir sekojoša: vispirms ir aplūkotas pasaules akciju indeksu multifraktalitātes pamati un konstruēts aktīvu ienesīguma multifraktāļu modelis, otrajā solī ir izklāstīta sākotnējo datu apstrādes procedūra, pēc tam tiek aplūkoti divi veivlet pārveidojumu moduļu maksimumu pieejas aspekti (veivlet analīze un multifraktāļu algoritms). Pēc tam veivlet pārveidojumu moduļu maksimumu pieeja ir izklāstīta saistībā ar pasaules akciju indeksiem uz Vācijas DAX30 akciju indeksa piemēra. Par eksperimenta objektiem ir kļuvuši 12 pasaules akciju indeksi: IBEX35 index, DAX30 index, Swiss Market Index, CAC40 index, FTSE100 index, Dow Jones Industrial index, Amsterdam Exchange index, Hang Seng index, NIKKEI225 index, Straits Times Index {Singapore}, Philippines Stock Exchange Index, BSE India Sensex 30 Index. Indeksi ir analizēti par pēdējiem 20 gadiem. Pēc akciju indeksu multifraktāļu analīzes rezultātiem, visi pētāmie objekti demonstrē stingri izteiktu multifraktāļu uzvedību, kas norāda uz dažādu fraktāļu mēru esamību. Pētījumā tika atklāti tipiskie fraktāļu mēri, kā arī multifraktāļu spektru nobīdes. Tāpat ir izpētīta akciju indeksu multifraktāļu spektru korelācija. Pētāmā metode ļauj atklāt indeksu līdzīgu uzvedību, kas nozīmē spēju fiksēt, turēt un glabāt tirgus informāciju "noteiktā veidā".

Андрей Пучков, Андрей Матвеев. Метод модулей максимумов вейвлет коэффициентов для мультифрактального анализа фондовых индексов
Данная статья призвана определить лучший подход, который будет способен выявить разницу в мультифрактальных спектрах различных биржевых индексов. Данный подход должен быть применим для мультифрактального анализа и имитационного моделирования азиатских и европейских биржевых индексов. Мультифрактальный анализ осуществлён с использованием так называемого метода модулей максимумов вейвлет коэффициентов (Wavelet Transform Modulus Maxima), который включает в себя два основных аспекта: вейвлет анализ (прямое непрерывное вейвлет преобразование и построение скелетона) и мультифрактальный алгоритм (функция обобщенной статистической суммы, функция масштабирования, функция мультифрактального спектра). Метод модулей максимумов вейвлет коэффициентов является методом для определения фрактальной размерности сигнала. В статье изложение результатов осуществляется следующим образом: прежде всего, освещаются основы мультифрактальности биржевых индексов, строится модель мультифрактальной доходности активов, после чего излагается процедура начальной обработки биржевых индексов, затем освещаются оба аспекта метода модулей максимумов вейвлет коэффициентов: вейвлет анализ и мультифрактальный алгоритм. После этого упомянутый метод иллюстрируется применительно к мультифрактальному анализу биржевых индексов на примере немецкого индекса DAX30. Затем описывается само исследование мультифрактального анализа биржевых индексов и его результаты. Объектами исследования являются 12 мировых биржевых индексов: IBEX35 index, DAX30 index, Swiss Market Index, CAC40 index, FTSE100 index, Dow Jones Industrial index, Amsterdam Exchange index, Hang Seng index, NIKKEI225 index, Straits Times Index {Singapore}, Philippines Stock Exchange Index, BSE India Sensex 30 Index. Индексы проанализированы за последние 20 лет. Согласно результатам мультифрактального анализа, все анализируемые биржевые индексы демонстрируют мультифрактальное поведение, что означает наличие различных фрактальных размерностей одновременно. В исследовании были выявлены как типичные значения мультифрактального спектра, так и его размытость. Также проведен корреляционный анализ мультифрактальных спектров; с помощью разработанного метода можно обнаружить биржевые индексы с похожим поведением, что подразумевает их способность получать, обрабатывать и хранить информацию строго определенным образом.