World Stock Indexes Wavelet Coefficients Probability Distribution Research and Analysis

Andrejs Puckovs[1], Andrejs Matvejevs[2], [1-2]*Riga Technical University*

Abstract **– This paper describes capabilities of wavelets for financial time series analysis. In current research wavelet analysis is provided by using Continuous Wavelet Transform. For basic assumption of time series behavior is used so called Fractal Brownian Motion, which is general case of classical Brownian motion, than implies time series long-term memory behavior. In terms of wavelets this analysis is done by filtering financial time series that are playing the role of signal with filter, by using Gausian mother wavelet function. Saying absolutely precisely, Continuous Wavelet Transform is done in frequency domain by using Fourier images of both – Fractal Brownian motion process upper interval bound and Gausian mother wavelet functions. The stock index analysis is completed in terms of Fractal Brownian Motion, for specified parts of the process. The devision of process in smaller parts is done by using probability bands. Each part of the process was analyzed by using modified R/S analysis. R/S indicator fitting to FBM bound is made using least square error criteria in time domain. Similar fitting is made by using wavelet images which are the result of Direct Continuous Wavelet Transform.**

Keywords **– Wavelet analysis, Time series, Stock indexes, Stock Market Crisis, Fractal Brownian Motion, Continuous Wavelet Transform, Wavelet image, Fourier transform, mother wavelet function, Gausian wavelet function, R/S analysis, Local Scaling Exponent, Curve Fitting, Least Squares Fitting.**

I. INTRODUCTION

The term of Fractal Brownian motion goes to Fractal, Chaos and Dynamic systems, stochastic processes area The idea of use of so called Hurst exponent goes from Harold Edwin Hurst long-term dependence describing method developed in early 1950-th which is called R/S method, when for the object of research became Nile river discharges. Harold Edwin Hurst research is based of comparing the reach of time series to its standard deviation that brings fundamental scalability meaning of so called Hurst exponent indicator. In current research, R/S method is being used, but with some modifications, which are described later.

Since early 90-th, with the development of computation technologies, for very precious tool of analysis has become wavelet analysis and the theory of wavelets, developed by Haar, van Ness, Mandell, Rose, Daubeshies, which can be considered as a part significant of signal theory.

Currently such wavelet techniques as Discrete wavelet transform, Dyadic Wavelet transform, Continuous Wavelet transform are used in many technical applications, e.g. JPEG image compression technique (according JPEG 2000 standard) is based on wavelet multire solution analysis.

Development of wavelets theory has become a necessity by understanding of incompleteness in Fourier transform. First of all Fourier transform is not optimal dealing with finite signals, secondly it does not solve the problem of uncertainty in time and frequency domains. The idea of wavelets solves this problem, at least particularly. Wavelets has taken opportunities of signal representation by using orthogonal functions, which have serious analogy with vector representation in multiple dimensions simultaneously. Vectors are keeping an idea of signal representation in multiple (independent) dimensions simultaneously.

Representation of financial time series in both frequency and time domain is very beneficial to select signal of certain frequency in certain time interval. Selecting the financial time series in certain frequency interval keeps an idea of risk measure analysis, while selecting financial time series in certain time interval keeps an idea of ‘risk measure evolution’ analysis. In Continuous Wavelet Transform the role filter which is illuminating ‘measure’ of risk is playing shifted and scaled mother wavelet function. Mother wavelet function shifting operation means illumination of risks in certain time interval, while scaling operation means illumination of risk measure (the higher risk is illuminated by lower scaling parameter while lower risk is illuminated by higher scaling parameter, saying absolutely precisely the measure of risk is calculated as inner product of time series and (scaled and shifted) mother wavelet function, and ‘lower risk’ is detected by using higher scaling parameter, but ‘higher risk’ is detected by using lower scaling parameter).

In other words the idea of independence is used in Continuous Wavelet Transform as idea of projection of the signal on different orthogonal mother wavelet functions, which are illuminating different frequencies in different time intervals. This way the realization of signal representation in both domains is done in Continuous Wavelet Transform technique. But as it is mentioned before, the wavelet theory has solved the problem of uncertainty in time and frequency domain only particularly. The nature of inability to represent signal in both domains simultaneously is also being seen in inability to illuminate ‘higher risk components’ without having enough information in time domain, that’s why wavelet coefficients at highest scaling parameters and larger shifting parameters does not have sense.

In terms of stochastic process analysis Continuous Wavelet transform has a meaning of decomposition of stochastic signal to smaller parts, each of them is representing ‘noise’ component in selected region of frequency. The selection of region of frequency is done by calculating inner product of stochastic signal (in current case- time series). By doing Direct and Inverse Continuous Wavelet Transform, wavelet analysis should decompose selected stochastic signal in small parts, each of them having standard deviation in particular time. In terms of Direct Continuous Wavelet transform standard deviation of stochastic process has a meaning of amplitude of 'noise' of certain frequency region. In terms of Inverse Continuous Wavelet transform amplitude of 'noise' of certain frequency region has a meaning of standard deviation.

II. Fractal Brownian Motion

Fractal Brownian motion is a generalized Classical Brownian motion, which is extended process by 'adding' additional parameter H – hurts exponent, which is Classical Brownian motion case with $H = 0.5$. See the next equation.

$$P(\Delta X < x) = (2\pi)^{\frac{-1}{2}} \cdot \sigma \cdot \Delta t^{H} \cdot \left. \cdot \int_{-\infty}^{x} \exp\left(\frac{-1}{2}\left(\frac{u}{\sigma \cdot (\Delta t)^{H}}\right)\right) du \right)\Bigg] \quad (1)$$

Where:

ΔX - increase of stochastic process X by Δt time units;

σ - standard deviation of stochastic process;

Δt - increase in time units;

H - Hurst exponent.

Stochastic process can be rewritten in terms of zero-mean normal distribution function.

$$P(\Delta X < x) = \Phi(x, 0, \sigma \cdot \Delta t^{H}). \quad (2)$$

Where:

$\Phi(x, \mu, \sigma)$ - Normal distribution function of random variable x, with mathematical expectation μ and standard deviation σ.

In terms of Fractal Brownian Motion process upper and lower bound can be scripted via following equation.

$$X_{ext} = k(P) \cdot \sigma \cdot \Delta t^{H}. \quad (3.1)$$

$$k(P) = \Phi_{inv}(P), \Phi_{inv}(\Phi(P, 0, 1) = X) = P, (0, 1). \quad (3.2)$$

Where:

X_{ext} - Fractal Brownian motion process upper (if $P > 0.5$) and lower (if $P < 0.5$) bound;

$k(P)$ - Fractal Brownian motion process upper/lower bound is constant, independent from σ, Δt, H, and dependent only from P (probability criteria of upper/lower bound).

Since P is chosen, obviously there is no problem to define intervals where the process 'lies' with a certain probability P, for example if denote that $0.05 < P < 0.95$, we can estimate two constants for lower $k(0.05) = \Phi_{inv}(0.05, 0, 1) \approx -1.6449$ and upper $k(0.95) = \Phi_{inv}(0.95, 0, 1) \approx +1.6449$ interval bounds.

III. Continuous Wavelet Transform

Continuous Wavelet Transform is powerful wavelet tool for signal analysis and synthesis. Since Direct Continuous Wavelet Transform is an operation of signal decomposition into a wavelet matrix, representing coefficients of inner product of original signal by shifted and scaled mother wavelet function, Inverse Continuous Wavelet Transform operates in synthesis side. Should say that Inverse Continuous Wavelet Transform operation should require admissibility condition, and not for all mother wavelet functions the signal can be retained by its components.

As it is mentioned before Direct Continuous Wavelet Transform is keeping an idea of signal representation by using shifted and scaled mother wavelet function, that by shifting operation selects signal parts in certain region of time, but by using scaling operation, defines desired frequencies of the signal indirectly, and selects signal part of desired region of frequency and in particular time interval, therefore realize an idea of signal representation in both: frequency and time domains.

Direct Continuous Wavelet Transform in terms of stochastic signal representation decompose original stochastic signal in its smaller parts. The signal decomposition is made by using different scaling parameters. Indirectly larger scaling parameter 'selects' lower frequency signal, that mostly is having smoothed form and lower standard deviation in terms of stochastic processes, while lower scaling parameter 'selects' high frequency signal, that mostly is having sharp form and higher standard deviation in terms of stochastic processes.

Direct Continuous Wavelet Transform in terms of Fourier transform can be found by using following equation:

$$CWT(a, \tau) = \int_{-\infty}^{\infty} \hat{X}(w) \cdot \hat{\Psi}(a \cdot w) \cdot \sqrt{a} \cdot e^{jw\tau} dw. \quad (4)$$

Where:

$CWT(a, \tau)$ - a Continuous Wavelet Transform with scaling parameter a and shifting parameter τ;

$\hat{X}(w)$ - a Fourier image of $X_{ext}(\Delta t)$ function (for conveniences lets rewrite $X_{ext} \to x(t), \Delta t \to t, \hat{X}(w) \xleftarrow{FT} x(t)$;

$\hat{\Psi}(a \cdot w)$ - a Fourier image of mother wavelet function $\psi(t)$ function.

For case specified the Gausian wavelet function is chosen; the choice is made because this function has very simple analytic form, see the next equation:

$$\psi(t) = e^{\frac{-t^2}{2}}. \quad (5)$$

Where:

$\psi(t)$ - Gausian wavelet function defined in time domain.

Since mother wavelet function and signal (that implies upper/lower bound function in time domain) are defined, the following function representations in frequency domain can be calculated, first signal function is represented:

$$x(t) \xrightarrow{FT} \hat{X}(w) = \int_{-\infty}^{\infty} k \cdot \sigma \cdot \Delta t^H \cdot e^{-jwt} dt. \quad (6)$$

$$\hat{X}(w) = -\Gamma(h+1, jtw) \cdot k \cdot \sigma \cdot t^{h+1} \cdot (jtw)^{-h-1}. \quad (7)$$

$$\Gamma(A,Z) = \int_{Z}^{\infty} t^{A-1} \cdot e^{-T} dT. \quad (8)$$

Where:

$\Gamma(A,Z)$ - under defined incomplete Gamma function

Analogically Fourier image of Gausian mother wavelet function could be estimated:

$$\psi(t) \xrightarrow{FT} \hat{\Psi}(w) = \int_{-\infty}^{\infty} e^{\frac{-t^2}{2}} \cdot e^{-jwt} dt. \quad (9)$$

$$\hat{\Psi}(wa) = \sqrt{\frac{\pi}{2}} \cdot e^{\frac{-a^2 w^2}{2}} \cdot erf\left(\frac{jaw}{\sqrt{2}} + \frac{t}{\sqrt{2}}\right). \quad (10)$$

$$erf(A) = \frac{2}{\sqrt{\pi}} \cdot \int_{0}^{A} e^{-T^2} dT. \quad (11)$$

Therefore $CWT(a,\tau)$ could be estimated in following way:

$$CWT(a,\tau) = \cdot\sqrt{\frac{\pi \cdot a}{2}} \cdot k \cdot \sigma \cdot t^{h+1} \cdot \int_{-\infty}^{\infty} \Gamma(h+1, jtw) \cdot (jtw)^{-h-1} \cdot \left. \cdot e^{j\tau w - \frac{a^2 w^2}{2}} \cdot erf\left(\frac{jaw}{\sqrt{2}} + \frac{t}{\sqrt{2}}\right) \right) dw \Big] \quad (12)$$

Provided algorithm explains Fractal Brownian Motion upper/lower bound behavior by using shifted and scaled Gausian mother wavelet function, which has most simple analytical form. Expect computational analysis should bring more light on calculation results.

IV. Stock Index Research

In current research financial time series are analyzed. Objects of research are worldwide stock indexes. Stock index data is beneficial for research because it shows behavior of various stock markets in different regions and includes only general information of stock market and does not include too much specific information about underling industries or companies. World stock indexes are liquid, usually no gaps in data excluding holidays is presented. In current research world stock indexes daily close prices from 1. Jan 2000 to nowadays are used.

World stock indexes are analyzed in terms of stochastic processes. For process representation the reach indicator is being used. Saying absolutely precisely, for current research the R/S indicator is being used, when the process reach is normalized by process standard deviation, which is the most classical method being used for Hurst exponent estimations. But in current research some R/S modifications are made, which are described next.

The general idea of research is to compare the reach indictor to correspondent theoretical Fractal Brownian motion (FBM) upper/lower bound interval – the process reach function of time is fit to FBM upper/lower bound function of time by changing underlying FBM parameters described in equation 3.1. For FBM estimation the least squared error criteria is used.

Since the general idea can be realized for the whole process data, it can be realized also for filtered data. In other words in current research data filtering is being used. The criteria of filtering proposed are interval bounds, which are calculated from process probability distribution.

The idea of filtering comes from idea of multifractality. The main questions to be answered is: "Does the Hurst exponent indicator is representative for all process, or some parts of process with different Hurst exponent can be identified, is the behavior of different parts of process the same?"

Identification of process parts should be reliable. Since the process is contains stock index data, which include 'stock market crisis times' and 'calm times' it would be wise to split the stock index data by using stock index increase filtering by interval bounds, in other words to split the data in percentiles and analyze splited data separately. If split parts do not vary in their behavior significantly the split has no meaning and data should be analyzed in together.

In current research the data are splited in intervals; the splited data are analyzed separately by using R/S method which is described in details next. In this case Hurst exponent is not appropriate term – since we are speaking not about whole process but about part of it, Hurst exponent should be replaced with local scaling exponent h.

The core idea of represented R/S analysis is to compare 'normalized reach indicator' which is a function of time, to correspondent FBM upper/lower bound, which also is a function of time. In other words FBM upper/lower bound (lets call FBM 'curve') should be fit to R/S indicator by changing specified parameters of FBM, which are described in equation 3.1. One of them is parameter $k(P)$ which is a function of P. It is intuitive clear that selected $k(P)$ parameter should be related to interval bounds which are used for filtration of process or spiting of process in different parts. The formal exploration of relation $k(P)$ parameter to FBM is described next.

Since R/S indicator fitting to FBM 'curve' can be made in time domain, the same operation can be made with wavelet images (which are Direct CWT result of underling R/S function and FBM 'curve' function represented in time domain).

The switch from functions represented in time domain to wavelet images should bring more flexibility in understanding of local scaling exponent h. First assume that there is not only one local scaling exponent h but the scope of local scaling exponents representing specified splited part of the process, which also vary dependently on wavelet scaling parameter $h = h(a)$. If local scaling exponent vary significantly depending on wavelet scaling parameter, local scaling exponent h has interpretation in terms of periodicity, otherwise it does not has meaning.

Fitting of wavelet image of FBM 'curve' to wavelet image of R/S function is done in a similar way as it done in time domain by using least squares error criteria. In terms of wavelet image fitting the FBM 'curve' R/S function and error function in time domain is transformed to wavelet image and its energy criteria is used for fitting. The algorithm is described next.

V. RESEARCH ALGORITHM

Here and further stock index research algorithm is described. The first step is obtaining financial time series data. After the correspondent index daily close prices are obtained, the data pre-processing step should be made.

$$X_{\log}(t) = \ln(X_c(t)), t = 1..T, t \in Z; \quad X_c(t) \leftarrow X_{\log}(t); \quad . \qquad (13)$$

For simplicity the new $X_{\log}(t)$ variables are let to older abbreviated $X_c(t)$ This is done in order to not use a lot of indexes.

After logarithm of the price data is taken, price increase should be found by using next equation.

$$\Delta X(t) = X_c(t) - X_c(t-1), t = 1..T-1, t \in Z . \qquad (14)$$

In the next step stock index increase $\Delta X(t)$ is normalized.

$$\Delta X_{norm}(t) = (\frac{\Delta X(t) - \mu}{\sigma}) . \qquad (15.1)$$

$$\mu = T^{-1}\int_1^T \Delta X(t)dt . \qquad (15.2)$$

$$\sigma = T^{-1}\int_1^T (\Delta X(t) - \mu)^2 dt \quad \Delta X(t) \leftarrow \Delta X_{norm}(t) \qquad (15.3)$$

In current normalization operation, normalization is done for all $\Delta X(t)$ variables. This step normalizes the process to $N(0,1)$ parameters. Normalization operation makes filtering operation possible, which is undertaken further. After normalization the change in variables is done: $\Delta X(t) \leftarrow \Delta X_{norm}(t)$.

In the next step $\Delta X(t)$ splits process in some parts to realize further research separately. See split realization in next equation.

$$\Delta X_{filtered}(i) = \Delta X(t) | \cdot (\Delta X_{\min} \geq \Delta X(t) > \Delta X_{\max})] \qquad (16.1)$$

$$\Delta X_{\min} = \Phi_{inv}(P_L, 0,1) . \qquad (16.2)$$

$$\Delta X_{\min} = \Phi_{inv}(P_H, 0,1) = \Phi_{inv}(P_L + \Delta P, 0,1) \qquad (16.3)$$

In current step $\Delta X(t)$ is assigned to $\Delta X_{filtered}(i)$ in case it satisfies $\Delta X_{\min} \geq \Delta X(t) > \Delta X_{\max}$, where $\Delta X_{\min}$ and $\Delta X_{\max}$ are interval lower and upper bounds, calculated via normal inverse distribution function $\Phi_{inv}(P,0,1)$. In such case bounds are defined by lower P_L and higher P_H probability variables, let's call it probability band.

Here and further filtered process $\Delta X_{filtered}(i)$ research is described. For convenience let's rewrite $\Delta X(t) \leftarrow \Delta X_{filtered}(i)$, taking on account we are operating with filtered data (another words with a part of the process). By analogy let's consider $t = 1..T, t \in Z$ and T is the last index of specified part of the process.

In the next step, normalization should be done, in this case, for a filtered data.

$$\Delta X_{norm}(t) = (\frac{\Delta X(t) - \mu}{\sigma}) . \quad (17.1)$$

$$\mu = T^{-1}\int_{1}^{T} \Delta X(t)dt . \quad (17.2)$$

$$\sigma = T^{-1}\int_{1}^{T} (\Delta X(t) - \mu)^2 dt \quad (17.3)$$

After normalization the change of variable $\Delta X(t) \leftarrow \Delta X_{norm}(t)$ is done.

In the next step the splited process part $X(t)$ is recovered by using underlying increase variables $\Delta X(t)$. See the next equation.

$$X(t) = \int_{1}^{t} \Delta X(u)du; . \quad (18)$$

The reach indicator is calculated by using $X(t)$ in next equation.

$$\begin{aligned} &Seq(t) = \{X(1), X(2),...,(X(t))\}, \\ &R(t) = 0.5\cdot(\max\{Seq(t)\} - \min\{Seq(t)\}), \end{aligned} . \quad (19)$$

The cumulative standard deviation S(t) can be found by equation scripted next.

$$S(t) = t^{-1}\int_{1}^{t} (\Delta X(u) - \mu(t))^2 du . \quad (20.1)$$

$$\mu(t) = t^{-1}\int_{1}^{t} \Delta X(u)du . \quad (20.2)$$

After this R/S indicator could be estimated by using classical Hurst method, see the next equation.

$$RS(t) = \frac{R(t)}{S(t)}, . \quad (21)$$

In next actions the R/S indicator. which is reach indicator normalized with standard deviation, is compared to correspondent FBM upper interval bound (which is a FBM ‘curve’), with parameters described in equation 3.1. The next equation describe an ‘etalon’ FBM curve.

$$RS_e(H) = k(P)\cdot\sigma\cdot t^H . \quad (22)$$

Current $k(P)$ parameter is a calculated via inverse normal distribution function by equation 16.3.

Since least square error criteria is selected for fitting, let’s write an approximation ‘error’ function in next equation.

$$\varepsilon(H) = \ln(\int_{1}^{T} (RS_w(H,t) - RS(t))^2 dt) . \quad (23)$$

For each H should be calculated $RS_e(H)$ in order to find such H, which provides $\varepsilon(H)$ minimum.

$$\begin{aligned} &\forall H : \varepsilon(H) \\ &H = H_{\min} : \varepsilon(H_{\min}) := \min(\varepsilon(H)) \end{aligned} . \quad (24)$$

By analogy in case of fitting of local scaling exponent $h(a)$ least square error is found in terms of wavelet image. For each local scaling exponent correspondent FBM ‘curve’ is constructed. FBM ‘curve’ is transformed to wavelet image by using CWT. By analogy using the same technique wavelet image is constructed for RS function.

Then for each scaling parameter a FBM ‘curve’ $RS_e(h,t)$ wavelet image W_e is compared to correspondent $RS(t)$ function curve image W_X. See the algorithm scripted in next equation.

$$\begin{aligned} &\forall h : RS_e(h,t) = k(P)\cdot\sigma\cdot t^h \\ &RS_e(t) \xrightarrow{DirectCWT} W_e(a,\tau),(RS_e(h,t)); \\ &RS(t) \xrightarrow{DirectCWT} W_X(a,\tau),(RS(t)); \\ &\forall a, a = 1,2,..A : \varepsilon_a(h) \\ &\varepsilon_a(h) = \log(\int_{1}^{A}\int_{1}^{T} (W_e(a,\tau,h) - W_X(a,\tau))^2 d\tau da) \end{aligned} .(25)$$

According least error criteria minimization task should find the best fitting parameter h for each a scaling parameter.

$$\begin{aligned} &\forall h : \varepsilon_a(h) \\ &h = h_{min} : \varepsilon_a(h_{\min}) := \min(\varepsilon_a(h)) \end{aligned} . \quad (26)$$

Provided algorithm is illustrated for stock index research next.

VI. RESEARCH EXAMPLE

Here and further provided algorithm is illustrated on example of Japan Stock Exchange Nikkei 225 index, which is considered in a period from 1985 Jan 1 to nowadays. An algorithm is written in Matlab code form.

```
% Loading the data
load Datafile.mat
```

```
% Signal preprocessing
Data =log(Data);
wn = diff(Data);

% Time space and std
s = 1;
T = length(Data); t = 1:T;

% Normalization
wn = (wn – mean(wn))./std(wn);
```

Nikkei 225 index after algorithm realization is illustrated next.

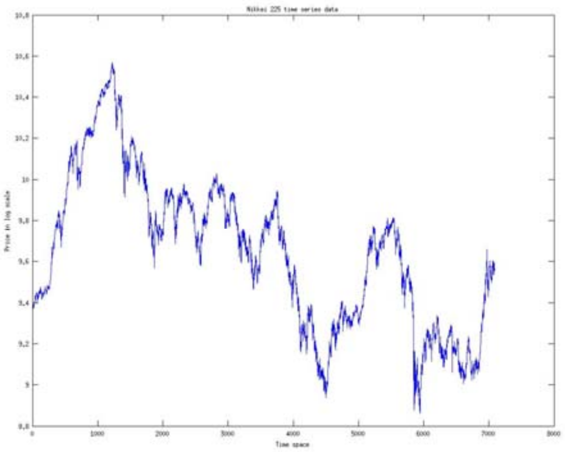

Fig. 1. Nikkei 225 Index data in log scale

Filtering operation or in other words operation of process splitting in parts is scripted in following way.

```
% Loading data
% Filtering:

P_m = linspace(0+sqrt(eps),1-sqrt(eps),100);

for pind = 1:length(P_m)-1
    Pl = P_m(pind);     x_l = norminv(Pl, 0, s);
    Ph = P_m(pind+1);   x_h = norminv(Ph, 0, s);
    Pd(pind) = 0.5*(Pl+Ph);

    %Spliting in intervals
    xind = find( wn>=x_l & wn<x_h );x = wn(xind);

    %OTHER CODE PLACED HERE IN FOR LOOP
end
```

In this code lower and upper bounds are defined, after this the process of specified bounds is selected.

One remark should be made here – all code considered next should be placed in specified place in for loop!

Normalization within filtered process is realized in following code.

```
% Normalization 2
x = (x – mean(x))./std(x);
```

R/S indicator is calculated by using following code:

```
X = cumsum(x);

% R/S calculation
R = zeros(size(X)); S = R';
    for i = 2:length(X)
        arg = X(1:i);
        darg = x(1:i);
        Lo(i) = min(arg);
        Up(i) = max(arg);
        S(i) = std(darg);
        clear arg darg
    end

R = 0.5 *(Up - Lo); RS = R./S; RS(1) = 0;
```

Constructed R/S indicator, which is a process reach indicator, normalized with its standard deviation, had to be compared to correspondent FBM 'curve' with appropriate process parameters. Fitting is realized by code scripted next:

```
% Fitting in 2D
t_e = 1:length(RS);
H_m = linspace(0,1,100);

for ind = 1:length(H_m)
        H = H_m(ind);
        RS_e = abs(x_h) * s * t_e.^H; %FBM 'curve'
        Err = RS_e - RS;
        Er(ind) = log(sum(Err.^2));
end

h(pind) = H_m(Er == min(Er));
```

Fitting results are represented in the next figure.

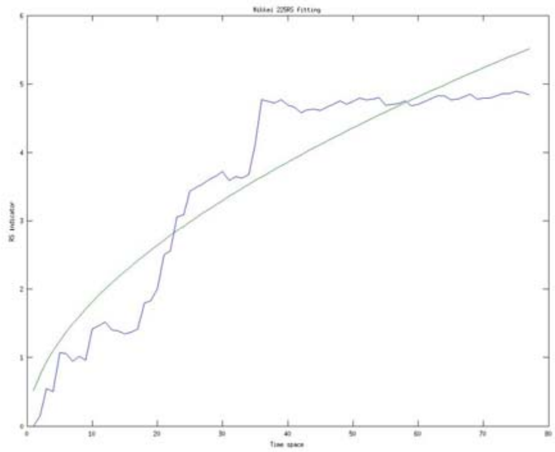

Fig. 2. Nikkei 225 process fitting in FBM curve

Same operation of fitting could me made by using R/S indicator and FBM curve wavelet images. At first wavelet image of R/S indicator is obtained by using following script

```
% Wavelet image construction (CWT of RS)
A = floor(T/4);
a = 1:A;
W = cwt(RS, a, 'mexh');
```

Fitting results are represented in the next figure .Fig. 1. Nikkei 225 process fitting in FBM curve

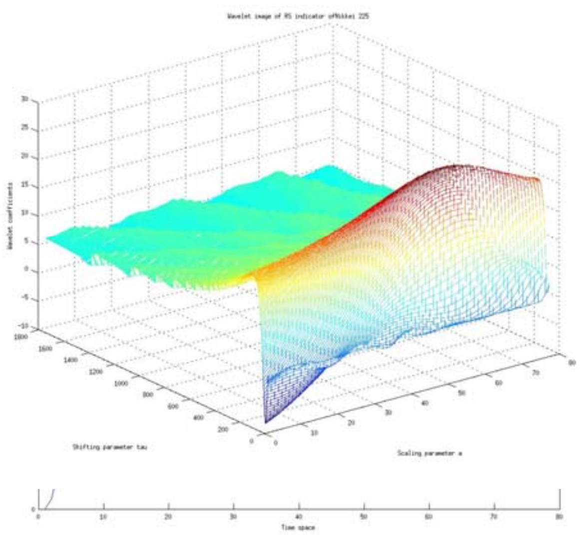

Fig. 3. A wavelet image of Nikkei 225 Index RS indicator

The same operation of fitting could me made by using R/S indicator and FBM curve wavelet images. At first wavelet image of R/S indicator is obtained by using following script.

Construction of wavelet image of FBM curve for each local scaling exponent h and scaling parameter a, error estimation and minimization is done by following code.

```
for aa = 1:A
    erp = inf;
    ind = 1;
    step = 10;

    while ind < length(H_m)
        H = H_m(ind);
        RS_e = abs(x_h) * s * t_e.^H;
            WRS_e = cwt(RS_e, aa, 'mexh');
            WEra = WRS_e-W(aa,:);
            er = log(sum(WEra.^2));
        if (ind+step>length(H_m)) && (step > 1)
            step = 1;
        elseif (ind+step>length(H_m)) && (step == 1)
            break
        end
        if (er<erp) && (step == 10)
            ind = ind+step;
            erp = er;
        elseif  (er>=erp) && (step == 10)
            ind = ind - step;
            step = 5;
            ind = ind + step;
        elseif (er>=erp)  && (step == 5)
ind = ind - step;
            step = 1;
            ind = ind + step;
        elseif (er>=erp) && (step == 1)
            break
        elseif (er<erp) && (step == 1)
            ind = ind+step;
            erp = er;
        else
            H_m(ind) = Hcom(pind);
            break
        end
    end

h_a(pind,aa) = H_m(ind);
end
```

Since wavelet image construction by using Direct CWT requires a lot of machine time, it’s is beneficial to check squared error ε_a for only specified h local scaling exponents and check the dynamics of ε_a. By increasing h parameter, squared error ε_a should decrease to minimum.

After fitting correspondent FBM 'curve' wavelet image is illustrated in next figure.

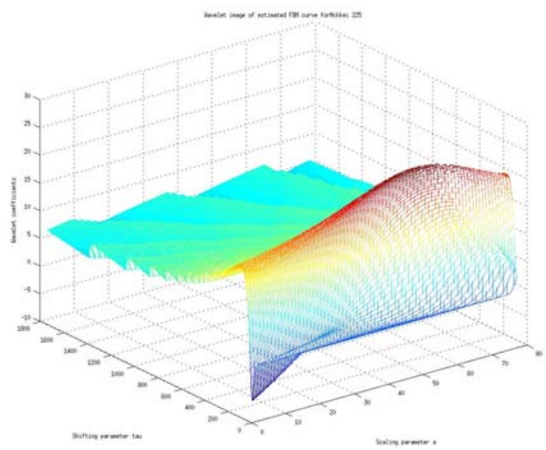

Fig. 4. A wavelet image of FBM curve estimated for Nikkei 225 Index

As it is shown here, visually fig.4 look similar to fig.3, as FBM 'curve' approximates R/S indicator.

Taking mentioned procedure step by step for each probability band Nikkei 225 local scaling exponents h are obtained. Analogically Nikkei 225 local scaling exponents $h(a)$ are obtained for each scaling parameter a for each probability band.

Nikkei 225 local scaling exponent h estimation results are shown in the next figure.

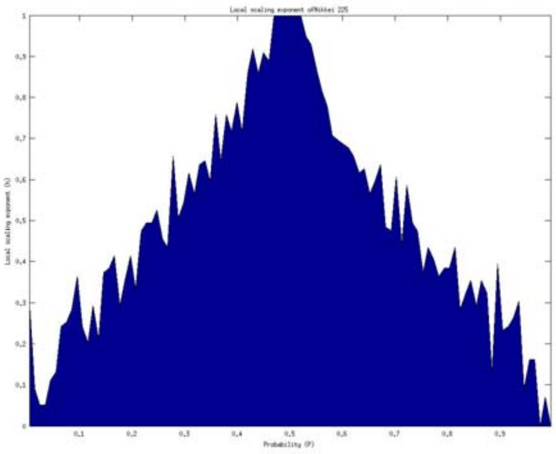

Fig. 5. Nikkei 225 local scaling exponent estimation, 2D plot

For convenience let's call figure 5 a multifractal spectrum of Nikkei 225 index.

As it is sown in figure 5, local scaling exponent h vary in each probability band significantly. At probability bands $P \approx 0.5$ local scaling exponent h reaches it maximum , but towards the edges local scaling exponent h is lower. Obviously the figure has left asymmetry, and some peak is observed in probability band $P \approx 0$. This is very interesting property, which is also described in other stock indexes. The exploration of this phenomenon is provided further.

As local scaling exponent h is estimated in terms of wavelet image for each a scaling parameter for each probability band, flowing estimation results are observed, see the next figure.

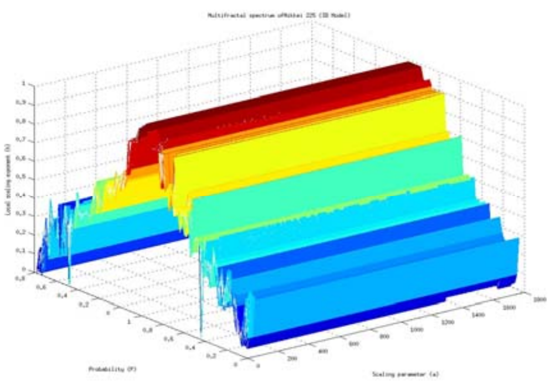

Fig.6 . Nikkei 225 local scaling exponent estimation, 3D plot

Nikkei 225 index local scaling exponent estimation in 3D has shown that local scaling exponent in current probability band does not vary significantly for all a scaling parameters. Despite this, some changes in a 'depth' occur, and for some scaling parameters a the local scaling exponent h is greater, but the changes are not significant. This phenomenon is illustrated in next plot.

Nikkei 225 local scaling exponent h estimation results are shown in the next figure.

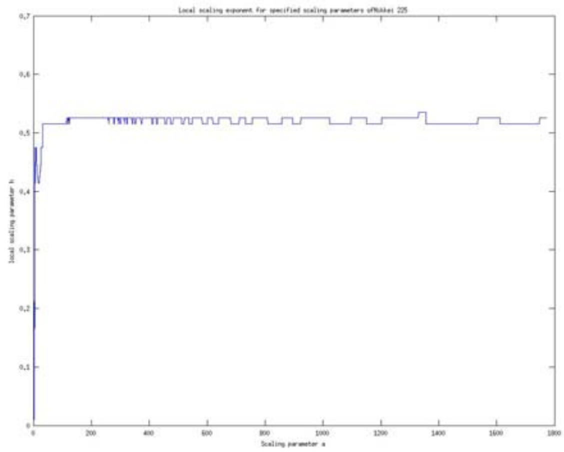

Fig. 7. Local scaling exponent estimation in scaling parameter depth

Changes in local scaling exponent h changes depending on scaling parameters a implies the presence of periodicity of local scaling exponent h , that assume some stable periods of stock index. If this is true, there should be some optimal investment horizons.

VII. Research results and discussions

In current research were analyzed following worldwide stock indexes - The Dow Jones Industrial Average, Amsterdam Stock Exchange, DAX30, IBEX 35, Hang-Seng Index, S&P BSE SENSEX' Nikkei 225, CAC 40.

Each of mentioned indexes was analyzed in similar way like Nikkei 225. Local scaling exponent estimation results for all of them are 'quite' similar. See illustrated estimation results in next figure.

Nikkei 225 local scaling exponent estimation results are shown in the next figure.

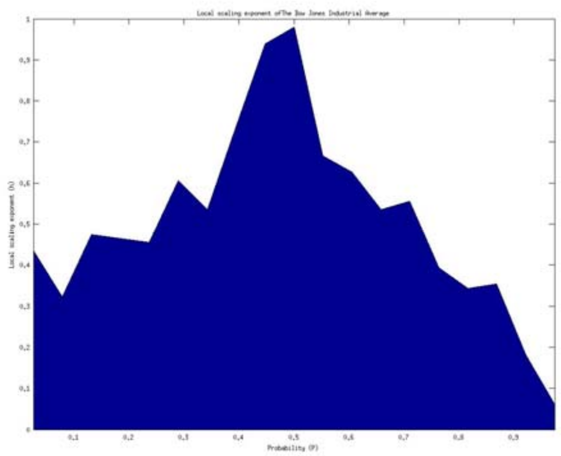

Fig. 7a. The Dow Jones Industrial local scaling exponent estimation, 2D plot

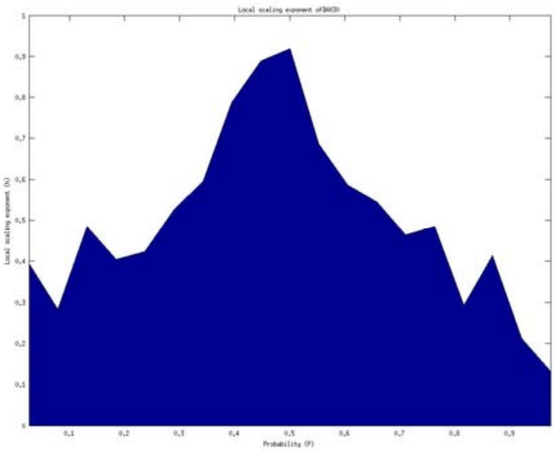

Fig. 7b. DAX30 local scaling exponent estimation, 2D plot

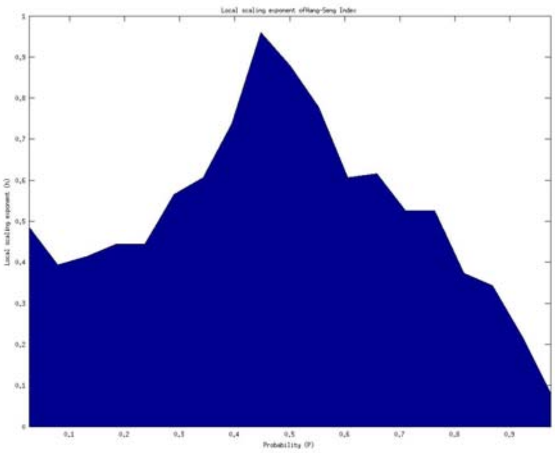

Fig. 7c. Hang Seng index local scaling exponent estimation, 2D plot

Looking at figure 7a, 7b and 7c, conclude all of indexes have own shape of multifractal spectrum's, on the other hand all figures obviouslv have left asymmetry – so in the probabilitv band $P \approx 0$ higher values of local scaling exponent h.

First of all this is a very specific property, which is not peculiar for Classical Brownian motion. In case of Brownian motion the value of local scaling exponent h decrease in edges to minimum (equallv on both sides).

The probability band $P \approx 0$ selects extremely low stock index parts, which actually are dramatic stock index dropdowns.

In terms of Fractal Brownian motion, the local scaling exponent h which in some sense is synonym of Hurst exponent shows the presence of persistent properties of time series. Higher local scaling exponent h is associated with a 'smooth', 'trendy' processes while lower scaling exponent h is associated with antipersistent properties, which usually mean the recovery to average.

In terms of stock markets higher local scaling exponent h for probability band $P \approx 0$ means that in case of significant dropdowns the backup or recovery to average is hardly possible. In case of dropdowns markets will remember the fail and negative trend will remain for sure.

On the other hand, in case of lucky the 'reversion' to average will be soon. while local scaling exponent h for probability band $P \approx 1$ is quite low and antipersistent properties should do the job and reversion to the mean will come soon.

Looking at figure 7a, 7b and 7c, sumarize local scaling exponent h values in the centre of figure are the highest. This interesting property show that in probability band $P \approx 0.5$ local scaling exponent h is high, consequently specified part of the process is 'smooth', 'trendy'. So this part of the process should be the most predictable.

References

[1] C. Chatfield, *The Analysis of Time Series*, NY: Champan&HALL/CRC, 2006, 329 pp. 255-268.

[2] V. M. Gadre, *Lecture series in Multirate Digital Signal Processing and Multiresolution analysis*. Bombay: IIT Bombay [Online] Available: http://www.youtube.com/watch?v=iNgV2NEyLhQ [Accessed 15. Okt 2013].

[3] T. K. Basu, Lecture Series on Digital Signal Processing. Kharagpur: IIT Kharagpur [Online] Available: http://www.youtube.com/watch?v=8zbBJkw5KsM

[4] http://www.yahoo.com. Amsterdam Exchange index data. [Online]. Available: http://finance.yahoo.com/q/hp?s=%5EAEX+Historical+Prices [Accessed 1. Okt 2013.]

[5] http://www.yahoo.com. BSE India Sensex 30 Index data. [Online]. Available: http:/finance.yahoo.com/q/hp?s=%5EBSESN+Historical+Prices [Accessed 1. Okt 2013.]

[6] http://www.yahoo.com. CAC40 index data. [Online]. Available: http://finance.yahoo.com/q/hp?s=%5EFCHI+Historical+Prices [Accessed 1. Okt 2013.]

[7] http://www.yahoo.com. DAX30 index data. [Online]. Available: http://finance.yahoo.com/q/hp?s=%5EGDAXI+Historical+Prices [Accessed 1. Okt 2013.]

[8] http://www.yahoo.com. Dow Jones Industrial index data. [Online]. Available: http://finance.yahoo.com/q/hp?s=%5EDJI+Historical+Prices [Accessed 1. Okt 2013.]

[9] http://www.yahoo.com. FTSE100 index data. [Online]. Available: http://finance.yahoo.com/q/hp?s=%5EFTSE+Historical+Prices [Accessed 1. Okt 2013.]

[10] http://www.yahoo.com. Hang Seng index data.[Online]. Available: http://finance.yahoo.com/q/hp?s=%5EHSI+Historical+Prices [Accessed 1. Okt 2013.]

[11] http://www.yahoo.com. IBEX35 index data.[Online]. Available: http://finance.yahoo.com/q/hp?s=%5EIBEX+Historical+Prices [Accessed 1. Okt 2013.]

[12] http://www.yahoo.com. NIKKEI225 index data.[Online]. Available: http://finance.yahoo.com/q/hp?s=%5EN225+Historical+Prices [Accessed 1. Okt 2013.]

[13] Wikipedia.org. Standard deviation, [Online]. Available: http://en.wikipedia.org/wiki/Standard_deviation. [Accessed 15.Okt 2013]

[14] Wikipedia.org.Percentile, [Online]. Available: http://en.wikipedia.org/wiki/Percentile [Accessed 15. Okt 2013]

[15] Кроновер, *Фракталы и хаос в динамических системах. Основы. теории.* М: Постмаркет, 2000. — 352 с.
[16] А. Н. Ширяев, Основы стохастической финансовой математики. М; Фазис, 1998 . – 983 стр.
[17] А. Н. Яковлев, *Введение в вейвлет преобразование.* Новосибирск: МИНОБР Новосибирский ГТУ, 2003, стр. 102.
[18] А. В. Новопашина, В. А. Саньков, В. Ю. Буддо, *Пространственно-временной анализ сейсмических структур байкальской рифтовой системы.* Иркутск: Институт земной коры СО РАН, 2011, 55-64.
[19] А. А. Пучков, *Анализ фрактального броуновского движения с помощью непрерывного вейвлет преобразования, с использованием вейвлет мексиканской шляпы.* Курск: Журнал научных публикаций аспирантов и докторантов. [Online]. Available: http://jurnal.org/articles/2013/mat5.html [Accessed 15. Okt 2013]

Andrejs Puckovs was born in Riga in 1985. He graduated from Riga Technical University (Latvia), where he received his Bachelor Degree in Management in 2007. He has the Qualification of Economist and Professional Master Degree in Economics (Financial analysis). Andrejs Puckovs is currently completing his research for a Doctoral Degree at Riga Technical University in the field of Mathematical Statistics and its Applications. From 2008 to 2010 he worked as a Fund Administrator in Custody and Correspondent Banking Department, JSC Swedbank. Contact data: Chair of the Probability Theory and Mathematical Statistics, Riga Technical University, email: rostock@inbox.lv.

Andrejs Matvejevs is a Doctor of Technical Sciences in Information Systems. Until 2009 he was a Chief Actuary at the insurance company "BALVA". For more than 25 years he has taught at Riga Technical University and Riga International College of Business Administration, Latvia. His previous research was devoted to solving of dynamical systems with random perturbation. His current professional research interests include applications of Markov chains to actuarial technologies: mathematics of finance and security portfolio. He is the author of about 40 scientific publications, two text books and numerous conference papers. Contact data: Chair of the Probability Theory and Mathematical Statistics, Riga Technical University, email:andrejs.matvejevs@rtu.lv

Andrejs Pučkovs, Andrejs Matvejevs. Pasaules akciju indeksu veivlet koeficientu varbūtību sadalījuma analīze
Šis raksts parāda finanšu laikrindu analīzes iespējas. Šajā rakstā veivlet analīze ir veikta, izmantojot Tiešo nepārtraukto viļņu pārveidojumu (Tiešo NVP). Rakstā ir izvirzīts pieņēmums par akciju indeksu uzvedību – par pamatu tika ņemta Fraktāļu Brauna kustība (FBK), kas ir klasiskās Brauna kustības vispārējs gadījums. Tika definēts FBK intervāla augšējā robeža, kas arīdzan tika analizēta, izmantojot veivlet analīzi. FBK intervālā augšējā robeža jeb FBK līkne ar Tiešo NVP tika pārveidota uz veivlet attēlu. Tiešais NVP tika veikts ar Furjē pārveidojumu. Furjē pārveidojums tika veikts gan FBK līknei, gan mātes veivlet funkcijai. Liela pētījuma daļa tika veltīta pasaules akciju indeksu analīzei. Akciju indeksu dati tika pārveidoti uz stohastisko procesu, kas tika sadalīts daļās. Sadalījums tika veikts, izmantojot varbūtību joslas. Katra procesa daļa tika analizēta, izmantojot R/S analīzi. Nosakot lokālo mērogošanas eksponenti, R/S rādītājs tika aproksimēts ar attiecīgo FBK līkni. Aproksimācijas kritērijs ir mazākā kvadrātiskā kļūda. Analoģiski R/S rādītāja jeb R/S funkcijas veivlet attēls tika aproksimēts ar FBK līknes veivlet attēlu.

Андрей Пучков, Андрей Матвеев. Анализ распределения вероятности вейвлет-коэффициентов мировых биржевых индексов.
Данная статья раскрывает возможности вейвлет-анализа финансовых рядов. В данной статье вейвлет анализ осуществлен с помощью Прямого Непрерывного Вейвлет-Преобразования (Прямого НВП). В данной статье основное предположение касательно поведения биржевых индексов сделано в пользу Фрактального Броуновского движения (ФБД); ФБД является общим случаем классического Броуновского движения. В ходе исследования определена верхняя граница интервала ФБД, которая стала объектом вейвлет-анализа. В ходе вейвлет-анализа верхняя граница интервал ФБД с помощью Прямого НВП преобразована в вейвлет-образ. Прямое НВП было выполнено посредствам преобразования Фурье-образа сигнала и материнского вейвлета. Основная часть работы посвещается анализу биржевых индексов. Данные биржерых индексов преобразованы в стохастический процесс, который разделен на части, каждая из которых была проанализирована по отдельности. Разделение процесса на части осуществлялась с помощью полос вероятности определенной ширены. Каждая часть процесса проанализирована посредством R/S анализа. Показатель размаха R/S аппроксимируется верхней границей интервал ФБД (или кривой ФБД) во временной области; в качестве критерия аппроксимации используется наименьшая квадратичная ошибка. Похожим образом аппроксимируется вейвлет образ показателя размаха R/S соответствующим вейвлет образом кривой ФБД. При этом делается допущение, что показатель ФБД (локальная скейлинговая экспонента) меняется на различных масштабах вейвлет-образа кривой ФБД.