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MEAN SQUARE LYAPUNOV EXPONENTS FOR LINEAR MARKOV SWITCHED DIFFERENCE EQUATIONS WITH NEAR TO CONSTANT COEFFICIENTS

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Abstract. The paper deals with the system of linear difference equations in $\mathbb{R}^d$ with the right part switched by homogeneous ergodic Markov chain $\{y_t, t \in \mathbb{Z}\}$ on the compact phase space $\mathbb{Y}$. We prove that there exists such the linear continuous operator $\mathbf{A}$ on the space of symmetric $d \times d$-matrix-function $\{q(y), y \in \mathbb{Y}\}$ that for any $t \in \mathbb{N}, y \in \mathbb{Y}$, and $x \in \mathbb{R}^d$ one can write the equality $\mathbf{E}\{(q(y_t)x_t, x_t)/y_0 = y, x_0 = x\} = \mathbf{E}\{(\mathbf{A}^t q)(y)x, x)\}$. This approach permits to derive such the basis matrix $\mathbf{B}(y)$ in the space $\mathbf{R}^d$ that the random process $z_t = B(y_t)x_t$ has the same mean square Lyapunov exponent as the solution of the above equation $x(t)$ and to propose convenient to application algorithm for asymptotic analysis of the equations with near to constant coefficients.

Key words: Markov difference equations, Markov switching, linear difference equation reducibility, Lyapunov exponent.

Mathematics Subject Classification: Primary 60H10, 60H30; Secondary 37H10.

1 Introduction

Our paper deals with the linear discrete dynamical system defined by the linear difference equation in $\mathbb{R}^d$

$$x_t = A(y_{t-1})x_{t-1}, \;\; t \in \mathbb{N} \tag{1}$$

where

- $\{A(y), y \in \mathbb{Y}\}$ – continuous $d \times d$-matrix fuction on the compact metric space $\mathbb{Y}$;
- $\{y_t, t \in \mathbb{Z}\} \subset \mathbb{Y}$ – given on filtered probabilistic space $(\Omega, \mathfrak{F}, \{\mathfrak{F}^t\}, \mathbf{P})$ $\mathfrak{F}^t$-adopted homogeneous Markov chain with the transition probability $p(y, dy)$ satisfying Feller property [5]:

$$\forall v \in \mathbb{C}(\mathbb{Y}): \;\; (\mathcal{P}v)(y) := \int_{\mathbb{Y}} v(z)p(y, dz) \in \mathbb{C}(\mathbb{Y}) \tag{2}$$

We will assume that above Markov chain satisfies the exponential ergodic property [5]: *spectrum* $\sigma(\mathcal{P})$ *of the operator* (2) *may be presented in a form:*

$$\sigma(\mathcal{P}) = \{1\} \bigcup \sigma_\gamma, \;\; \sigma_\gamma \subset \{\lambda \in \mathcal{C} : |\lambda| \leq \gamma < 1\} \tag{3}$$

with the simple spectrum poin 1. Owing this assumption there exists [5] an unique invariant probabilistic measure $\mu(dy)$, satisfying the equation

$$(\mathcal{P}^* \mu)(dz) := \int\limits_{\mathbb{Y}} \mu(dy) p(y, dz) = \mu(dz) \tag{4}$$

and for any distribution of the initial value y_0 the sequence of random variables y_t tends by probability to a random variable $\hat{y}$ with the distribution $\mu(dy)$. For any integers $t > s$ and any $x(s) = x \in \mathbb{R}^d$ $y(s) = y \in \mathbb{Y}$ one can define the solution of the equation (1) in a following form:

$$x(t, s, x, y) := X(t, s, y)x, \tag{5}$$

where

$$X(t, s, y) := A(y_{t-1})A(y_{t-2}) \cdots A(y_{s+1})A(y) \tag{6}$$

for $t > s$ and $X(s, s, y) \equiv I$. As it has been proved by many authors (see survay in [1]), any solution of the equation (1) has an exponential behavior and for any $x \in \mathbb{R}^d$ there exist the limits

$$\lambda(x) = \lim_{t \to \infty} \frac{1}{t} \ln |X(t, s, y)x| \tag{7}$$

and

$$\lambda^{(p)}(x) = \lim_{t \to \infty} \frac{1}{t} \ln \mathbf{E}\{|X(t, s, y)x|^p\}, \;\; \forall p \geq 0 \tag{8}$$

with $t \to \infty$ defines the set of *the Lyapunov exponents*:with $t \to \infty$ defines the set of *the Lyapunov exponents*: Besides under the ergodicity assumption (3) there exist the finite sets of real numbers $\mathcal{S} = \{\lambda_1, \lambda_2, ..., \lambda_m\}$ and $\mathcal{S}_p = \{\lambda_1^{(p)}, \lambda_2^{(p)}, ..., \lambda_n^{(p)}\}$ that $\lambda(x) \in \mathcal{S}$ and $\lambda^{(p)}(x) \in \mathcal{S}_p$ for any initial values $x(s) = x \in \mathbb{R}^d$ and $y(s) = y$. These sets are called *the Lyapunov spectrum* and *the Lyapunov* p*-spectrum*, the numbers $\lambda_j \in \mathcal{S}$ – *the Lyapunov exponents*, the numbers $\lambda_j^{(p)} \in \mathcal{S}_p$ – *the Lyapunov* p*-exponents*, , bet numbers $\lambda := \max\{\lambda_1, \lambda_2, ..., \lambda_m\}$ and $\lambda^{(p)} := \max\{\lambda_1^{(p)}, \lambda_2^{(p)}, ..., \lambda_n^{(p)}\}$ – *the Lyapunov index of stability* and *the Lyapunov* p*-index of stability* correspondingly. Our paper develops the method and algorithm for calculation of the numbers $\{\lambda_1^{(2)}, \lambda_2^{(2)}, ..., \lambda_n^{(2)}\}$ (called *the mean square Lyapunov exponents*) for the equation

$$x_t = A(y_{t-1}, \varepsilon)x_{t-1}, \;\; t \in \mathbb{N} \tag{9}$$

with near to constant matrix

$$A(y, \varepsilon) = A_0 + \sum_{k=1}^{m} \varepsilon^k A_k(y) \tag{10}$$

where $\varepsilon \in (0, \varepsilon_0)$ is a small parameter. So far as we are looking for an asymptotic approximation of the mean square Lyapunov exponenets we should deal with the mathematical expectation of the matrices $x(t)x^T(t), t \in \mathbb{N}$. Here and further an index “T” stands for transposition of a vector or a matrix. Writing the matrices $x(t)x^T(t)$ as the vector-columns $\mathbf{x}(t)$ we can derive the difference equation in the space $\mathbb{V} := \mathbb{R}^{d^2}$:

$$\mathbf{x}(t) = A(y_{t-1}, \varepsilon)x(t-1)x^T(t-1)A^T(y_{t-1}, \varepsilon) := \mathbf{A}(y_{t-1}, \varepsilon)\mathbf{x}(t-1) \tag{11}$$

It is obvious that the operators $\mathbf{A}(y, \varepsilon)$ in the above equation are the symmetric $d \times d$-matrix-functions of the parameters $y \in \mathbb{Y}$ and $\varepsilon > 0$. To derive the formulae for the Lyapunov mean square exponents we will apply our proposed the covariation semigroup method [2] and reducibility in the mean algorithm [3]. We will construct such the basis matrix functions $\mathbf{B}(y, \varepsilon)$ in the space $\mathbb{V}$ that the sequence $\mathbf{q}(t) = \mathbf{E}\{\mathbf{B}(y_{t+s}, \varepsilon)\mathbf{x}(t)/y_s = \hat{y}\}$ satisfies the finite demensional linear difference equation in the space $\mathbb{V}$

$$\mathbf{q}(t) = \Lambda(\varepsilon)\mathbf{q}(t-1) \tag{12}$$

We will prove that the bases matrices $\mathbf{B}(y, \varepsilon)$ and the matrix $\Lambda(\varepsilon)$ may be expanded in the series by powers of the paramter ε. This permits to derive the algorithm for construction of the matrix $\mathbf{L}(\varepsilon)$ with any degree of accuracy $O(\varepsilon)$. Not so difficult to prove that the set $\mathcal{S}_2 = \{\lambda_1^{(2)}, \lambda_2^{(2)}, ..., \lambda_n^{(2)}\}$ of the Lyapunov exponents for (12) may be determine as $\hat{\mathcal{S}}_2 = \{\rho = \ln|\lambda|, \lambda \in \sigma(\Lambda(\varepsilon)\}$ where $\sigma(\Lambda(\varepsilon)$ is the spectrum of the matrix $\Lambda(\varepsilon)$ (see, for example, [8] and [9]). By construction the matrices $\mathbf{B}(y, \varepsilon)$ are invertible and satisfy the inequalities:

$$\exists M > 0, \exists \varepsilon_0 > 0 : \ \max\{\sup_{y \in \mathbb{Y}, |\varepsilon| < \varepsilon_0} ||(\mathbf{B}(y, \varepsilon))^{-1}||, \sup_{y \in \mathbb{Y}, |\varepsilon| < \varepsilon_0} ||\mathbf{B}(y, \varepsilon)||\} < M < \infty \tag{13}$$

Because of inequalities

$$|\mathbf{x}(t)| = |\mathbf{B}(y, \varepsilon)(\mathbf{B}(y, \varepsilon))^{-1}\mathbf{x}(t)| \le M|\mathbf{B}(y, \varepsilon)\mathbf{x}(t)| \le M^2|\mathbf{x}(t)|$$

and

$$M^{-1}|\mathbf{E}\{\mathbf{B}(y, \varepsilon)\mathbf{x}(t)\}| \le |\mathbf{E}\{\mathbf{x}(t)\}| \le M|\mathbf{E}\{\mathbf{B}(y, \varepsilon)\mathbf{x}(t)\}|$$

we may be sure that the mean square Lyapunov spectrum for (1) coincides with the set $\hat{\mathcal{S}}_2$.

2 Reducibility method

2.1 Shift operator semigroup for conditional second moments

In concordance with our proposal reducibility method [3] we should define in the space $\mathbb{U}$ of d^2-dimensional continuous vector functions $\mathbf{u} = \{u(y), y \in \mathbb{Y}\}$ the linear continuous operator-family $\mathcal{L}(\varepsilon)$ using the equalities:

$$\mathbf{E}\{\mathbf{u}^T(y_{s+1})\mathbf{x}_{s+1}/y_s = y, \mathbf{x}_s = \mathbf{x}\} = (\mathcal{P}\mathbf{u}^T)(y)\mathbf{A}(y, \varepsilon)\mathbf{x} := (\mathcal{L}(\varepsilon)\mathbf{u})^T(y)\mathbf{x}$$

We extend this operator-family on the space $\mathbb{G}$ of the continuous $d^2 \times d^2$-matrix functions and $\mathbf{g} = \{g(y), y \in \mathbb{Y}\}$ by means of formula:

$$\mathbf{E}\{\mathbf{g}(y_{s+1})\mathbf{x}_{s+1}/y_s = y, \mathbf{x}_s = \mathbf{x}\} = (\mathcal{P}\mathbf{g})(y)\mathbf{A}(y, \varepsilon)\mathbf{x} := (\mathcal{L}(\varepsilon)\mathbf{g})(y)\mathbf{x}$$

that is

$$(\mathcal{L}(\varepsilon)\mathbf{g})(y) := (\mathcal{P}\mathbf{g})(y)\mathbf{A}(y, \varepsilon) = \left(\int_{\mathbb{Y}} g(z)p(y, dz)\right)\mathbf{A}(y, \varepsilon) \tag{14}$$

Applying this formula and the formula for a conditional expectation we can obtain the equality

$$\begin{aligned}\mathbf{E}\{g(y_{t+1})\mathbf{x}_{t+1}\} &= \mathbf{E}\{\mathbf{E}\{g(y_{t+1})\mathbf{x}_{t+1}/\mathbf{x}_t, y_t\}\} = \\ &= \mathbf{E}\left\{\mathbf{E}\left\{(\mathcal{P}\mathbf{g})(y_t)\mathbf{A}(y_t,\varepsilon)\mathbf{x}|_{\mathbf{x}=\mathbf{x}_t}\right\}/y_t\right\} = ((\mathcal{L}(\varepsilon)\mathbf{g})(y_t)\mathbf{E}\{\mathbf{x}_t/y_t\}\end{aligned}$$

Therefore for any initial values $x_s = x \in \mathbb{R}^d, y_s = y \in \mathbb{Y}$, and $\mathbf{g} \in \mathbb{G}$ we can iterate the above equality and derive the formula

$$\mathbf{E}\{g(y_{t+s})\mathbf{x}_{t+s})\} = ([\mathcal{L}(\varepsilon)]^t\mathbf{g})(y)\mathbf{x} \tag{15}$$

Any satisfying inequality $\min_{y\in\mathbb{Y}} \det b(y) = \beta > 0$ matrix $\mathbf{b} \in \mathbb{G}$ defines a basis in the space $\mathbb{V}$ and we can rewrite any vector $\mathbf{x} \in \mathbb{V}$ in new basis in a following form: $\mathbf{x} = (b(y)^{-1}\mathbf{z}$. Besides

$$\min_{y\in\mathbb{Y}}\{||b(y)||^{-1}\}|\mathbf{z}| \leq |\mathbf{x}| \leq \max_{y\in\mathbb{Y}}\{||(b(y))^{-1}||\}|\mathbf{z}| \tag{16}$$

If x_s is random variable with the matrix of the second moments

$$\mathbf{E}\{g(y_s)x_s x_s^T/y_s = y\} = \mathbf{E}\{g(y)\mathbf{x}_s/y_s = y\} := \mathbf{m}_s(y)$$

we can find from (15) the equation for conditional second momentt $\mathbf{m}_{t+s}(y) := E\{\mathbf{x}_{t+s}/y_s = y\}$

$$\mathbf{m}_{t+s}(y) := \mathbf{E}\{g(y_{t+s})\mathbf{x}_{t+s})\} = ([\mathcal{L}(\varepsilon)]^t\mathbf{g})(y)g^{-1}(y)\mathbf{m}_s(y) \tag{17}$$

for any $t > 0$. We will refere to $\{[\mathcal{L}(\varepsilon)]^t, t \in \mathbb{N}\}$ as *the shift operator semigroup for conditional second moments in the space* $\mathbb{V}$ *with the basis* $\{g(y), y \in \mathbb{Y}\}$ and to the operator $\mathcal{L}(\varepsilon)$ - *the generator of the shift operator semigroup*. We will take advantage of the opportunity to choose the basis in order to simplify the view of the formula (17). As it follows from (17) to find the mean square Lyapunov index for the equation (1) we should look for the maximal by modulus eigenvalue of the operator (14). This problem was argued in the paper [2]. Applying the results of the paper [7] the authors proved that there exists the positive simple eigenvalue $r(\varepsilon) \in \sigma(\mathcal{L}(\varepsilon))$ satisfying equality $r(\varepsilon) = \max\{|\lambda|, \lambda \in \sigma(\mathcal{L}(\varepsilon))\}$.

2.2 Reducibilty method

In theis section we construct the basis, which help us to derive reducibility algorithm for the equation (17). For that we will apply the linear operator perturbation theory stated in the monography [6]. Substituting (10) in (11) we can find the decomposition of the operator $\mathbf{A}(y, \varepsilon)$:

$$x \in \mathbb{R}^d, \mathbf{x} = xx^T : \mathbf{A}(y,\varepsilon)\mathbf{x} = \sum_{l=0}^{2m} \varepsilon^l \mathbf{A}_l(y)\mathbf{x} \tag{18}$$

where the $d^2 \times d^2$-matrix $\mathbf{A}_l(y)$ may be defined by equality

$$\mathbf{A}_l(y)\mathbf{x} = \left\{\sum_{k=0}^{l} A_k(y)xx^T A_{l-k}^T(y)\right\}, \quad l = 0, 1, 2, ..., 2m$$

after substitution the matrix xx^T as a d^2-dimensional vector-column $\mathbf{x}$. Applying the decomposition (18) we can find corresponding decomposition for the operator (14):

$$(\mathcal{L}(\varepsilon)\mathbf{g})(y) := \sum_{l=0}^{2m} \varepsilon_l(\mathcal{L}_l\mathbf{g})(y) \tag{19}$$

where

$$\mathbf{g} \in \mathbb{G}: \quad (\mathcal{L}_l \mathbf{g})(y) = \left(\int\limits_{\mathbb{Y}} g(z) p(y, dz) \right) \mathbf{A}_l(y) \tag{20}$$

By construction the linear continuous operator (20) is analytical function on parameter ε in the vicinity of the point $\varepsilon = 0$. Therefore [6] there exists such a positive number ε_0 that for all $\varepsilon \in \{|z| < \varepsilon_0)\}$ we may analyze the izolated spectrum points and the spectrum projectors of operator $\mathcal{L}(\varepsilon)$ making the most out of the spectral properties of the operator

$$(\mathcal{L}(0)\mathbf{g})(y) := \left(\int\limits_{\mathbb{Y}} g(z) p(y, dz) \right) \mathbf{A}_0 \tag{21}$$

If an isolated spectrum point $\lambda \in \sigma(\mathcal{L}(0))$ has the spectral subspace of dimension k with spectral projector $\mathbf{P}_\lambda$ then [6] the operators $\sigma(\mathcal{L}(\varepsilon))$ have such a set of the spectral points $\sigma_\lambda(\varepsilon) := \{\lambda_1(\varepsilon), \lambda_2(\varepsilon), ..., \lambda_l(\varepsilon)\}$ that $\lim\limits_{\varepsilon \to 0} \lambda_j(\varepsilon) = \lambda$ for each $j = 1, 2, .., l$. Corresponding to the spectral group $\sigma_\lambda(\varepsilon)$ the total spectral subspace has the same dimension k and the total spectral projector $\mathbf{P}_\lambda(\varepsilon)$ is analytically dependent on ε operator-function in some vicinity of the point $\varepsilon = 0$. This results we may apply to analysis of the operators $\mathcal{L}(\varepsilon)$ for sufficiently small $\varepsilon > 0$. By construction the operator (21) is the tensor product $\mathcal{L}(0) = \mathcal{P} \otimes A_0 \otimes A_0^T$ and so the spectrum $\sigma(\mathcal{L}(0))$ of this operator has the following form

$$\sigma(\mathcal{L}(0)) = \{\lambda_1\lambda_2\lambda_3, \lambda_1 \in \sigma(A_0), \lambda_2 \in \sigma(A_0), \lambda_3 \in \sigma(\mathcal{P})\} \tag{22}$$

Remind that spectrum $\sigma(\mathcal{P})$ satisfies the assumption (3). We assume also that

$$\{\lambda_1\lambda_2\lambda_3, \lambda_1 \in \sigma(A_0), \lambda_2 \in \sigma(A_0), \lambda_3 \in \sigma_\gamma)\} \cap \sigma_0 = \emptyset \tag{23}$$

where $\sigma_0 := \{\lambda_1\lambda_2, \lambda_1 \in \sigma(A_0), \lambda_2 \in \sigma(A_0)\}$. It is clearly to see that the operator $\mathcal{L}(0)$ leaves as invariant the space $\mathbb{G}_0 \subset \mathbb{G}$ of constant $d^2 \times d^2$-matrices and σ_0 is the spectral set of $\mathcal{L}(0)$ in this subspace. In concordance with Kato pereturbation theory [6] there exist such a positive number ε_0 that for $0 < \varepsilon < \varepsilon_0$ the operators $\mathcal{L}(\varepsilon)$ have the spectral sets σ_ε of the same dimension as σ_0 and the corresponding total projectors $\mathbf{P}(\varepsilon)$ may be presented in a form of the series

$$\mathbf{P}(\varepsilon) = \mathbf{I} + \varepsilon\mathbf{P}_1 + \varepsilon^2\mathbf{P}_2 + \cdots \tag{24}$$

Let $\mathbf{B}(\varepsilon) = \{B(y, \varepsilon), y \in \mathbb{Y}\}$ be the basis in the subspace $\mathbb{V}_\varepsilon := \mathbf{P}(\varepsilon)\mathbb{V}$, that is $d^2 \times d^2$-matrix with columns $\{\mathbf{b}_j, j = 1, 2, ..., d^2\}$. By definition of basis we can decompose the vector $\mathcal{L}(\varepsilon)\mathbf{b}_j$ as the linear combination

$$\mathcal{L}(\varepsilon)\mathbf{b}_j = \sum_{k=1}^{d^2} \lambda_{kj}\mathbf{b}_k$$

for each $j = 1, 2, ..., d^2$. This assertion leads to the matrix equation

$$\left(\int\limits_{\mathbb{Y}} B(z, \varepsilon) p(y, dz) \right) \mathbf{A}(y, \varepsilon) = \Lambda(\varepsilon) B(y, \varepsilon) \tag{25}$$

where $\Lambda(\varepsilon)$ and $B(y, e)$ are analytically dependent on parametra ε matrices in the vicinity of the point $\varepsilon = 0$. As it has been proved in the Introduction of our paper the mean square Lyapunov spectrum for our difference equation (1) is the set of the real numbers $S_2(\varepsilon) = \{|\lambda|, \lambda \in \sigma(\Lambda(\varepsilon))\}$ and the mean square Lyapunov index is the maximal by modulus a real spectrum point of the matrix $\mathcal{L}(\varepsilon)$.

2.3 Reducibility algorithm and the Lyapunov spectrum

Owing the analyticity of the matrix functions $\mathbf{B}(\varepsilon)$ and $\Lambda(\varepsilon)$ by ε in the some vicinity of the point $\varepsilon = 0$ we can substitute the series

$$B(y,\varepsilon) = B_0 + \sum_{k=1}^{\infty} \varepsilon^k B_k(y); \mathbf{A}(\varepsilon) = \mathbf{A}_0 + \sum_{k=1}^{\infty} \varepsilon^k \mathbf{A}_k; \Lambda(\varepsilon) = \Lambda_0 + \sum_{k=1}^{\infty} \varepsilon^k \Lambda_k \tag{26}$$

in the equation (25). Without loss of generality we may choose $d^2 \times d^2$-matrix unit I as the basis matrix in the space $\mathbf{P}(0)\mathbb{V} = \mathbb{R}^{d^2}$ and substitute in (26) $B_0 = I$. Equating the terms near the like powers of parameter ε in the equation (25) we will have the equality $\Lambda_0 = \mathbf{A}_0$ and the series of the matrix equations:

$$(\mathcal{P}\mathbf{B}_1)(y)\mathbf{A}_0 - \mathbf{A}_0 B_1(y) = \Lambda_1 - \mathbf{A}_1(y) \tag{27}$$

$$(\mathcal{P}\mathbf{B}_2)(y)\mathbf{A}_0 - \mathbf{A}_0 B_2(y) = \Lambda_2 + \Lambda_1 B_1(y) - (\mathcal{P}\mathbf{B}_1)(y)\mathbf{A}_1(y) - \mathbf{A}_2(y) \tag{28}$$

etc. Each of the above equations has a form of linear equation in the space $\mathbf{C}_{d^2}(\mathbb{Y})$ of the continuous $d^2 \times d^2$-matrix functions:

$$(\mathcal{D}\mathbf{B})(y) = F(y) \tag{29}$$

where the linear continuous operator $\mathcal{D}$ is defined by equality $(\mathcal{D}\mathbf{B})(y) = (\{\mathcal{P} \otimes \mathcal{H}\}\mathbf{B})(y)$, where $\mathcal{H}$ is the linear operator on the space of $d^2 \times d^2$ constant matrices $\mathcal{H}B = B\mathbf{A}_0 - \mathbf{A}_0 B$. The adjoint space $\mathbf{C}^*_{d^2}(\mathbb{Y})$ is the space of the count additive $d^2 \times d^2$-matrix measures [4] and the scalar product for $\mathbf{B} \in \mathbf{C}_{d^2}(\mathbb{Y})$ and $\mathbf{M} \in \mathbf{C}^*_{d^2}(\mathbb{Y})$ may be given by equality $< \mathbf{B}, \mathbf{M} >:= \int_{\mathbb{Y}} \mathrm{trace}\{B(y)M^T(dy)\}$. Defined by this scalar product the adjoint to $\mathcal{D}$ operator has a form

$$(\mathcal{D}^* M)(dy) = \mathbf{A}_0^T(\mathcal{P}^*\mathbf{M})(dy) - M(dy)\mathbf{A}_0^T = ((\mathcal{P}^* \otimes \mathcal{H}^*)\mathbf{M})(dy)$$

According to the Fredholm alternative [4] the equation (29) has a solution if and only if the right part of this equation is orthogonal to each solution of the equation $\mathcal{D}^* M = 0$. It is easy to prove that any solution $\mathbf{M}$ of the equation $\mathcal{D}^* M = 0$ may be given in a form of product μU where U is constant $d^2 \times d^2$-matrix satisfying the equation $\mathbf{A}_0^T U - U\mathbf{A}_0^T = 0$ and the measure μ is defined by formula (4). Therefore the equation (29) has a solution if and only if $\overline{F} := \int_{\mathbb{Y}} F(y)\mu(dy) = 0$. Remind that we are looking for some basis matrix $B(y,\varepsilon)$ satisfying equation $\max_y |\det B(y,\varepsilon)| > 0$ and therefore we have option for the matrices $B_j(y)$. Therefore we can choose such the matrices $\mathbf{B}_j, j \in \mathbb{N}$ that $\overline{\mathbf{B}}_j = 0$. The above assertions permit to chose $\Lambda_1 = \mathbf{A}_0$, to solve the equation (27), and to substitute $\mathbf{B}_1$ in the right part of the equation (28). Now we can apply the Fredholm alternative to (28) and find matrix Λ_2:

$$\Lambda_2 = \overline{\mathbf{A}_2(y) - \overline{\mathbf{A}_1} B_1(y) + (\mathcal{P}\mathbf{B}_1)(y)\mathbf{A}_1(y)} = \overline{\mathbf{A}}_2 + \overline{(\mathcal{P}\mathbf{B}_1)(y)\mathbf{A}_1(y)} \tag{30}$$

Next we should solve the equation (28) and substitute the matrix $\mathbf{B}_2$ into equation for the matrix $\mathbf{B}_3$ and etc.

2.4 Example

Let $\{y(t), t \in \mathbb{R}\}$ be the Markov chain on the phase space $\mathbb{Y} = \{-1, 1\}$ with transition probability matrix

$$\mathbf{P} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}, \quad p \in (0, 1/2)$$

and the two-dimensional vector-sequence $\{x_t, t \in \mathbb{N}\}$ satisfies the difference equation

$$x_t = A(y(t-1), \varepsilon)x_{t-1}, \quad A(y, \varepsilon) = \begin{pmatrix} 0 & 1 \\ 1 + \varepsilon y\beta & \varepsilon^2 y\alpha \end{pmatrix} \tag{31}$$

Now we should write the equation for the matrix $x_t x_t^T$. This symmetric matrix may be specified by the vector

$$\vec{X}(t) := \begin{pmatrix} x_1^2(t) \\ x_1(t)x_2(t) \\ x_2^2(t) \end{pmatrix}$$

and therefore we should deal with the equation

$$\vec{x}(t+1) = \mathbf{A}(y(t), \varepsilon)\vec{x}(t) \tag{32}$$

where

$$\mathbf{A}(y, \varepsilon) = \mathbf{A}_0 + \varepsilon y\mathbf{A}_1 + \varepsilon^2\{\mathbf{A}_{20} + y\mathbf{A}_{21}\} + \varepsilon^3\mathbf{A}_3 + \varepsilon^4\mathbf{A}_4$$

$$\mathbf{A}_0 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ 2\beta & 0 & 0 \end{pmatrix}; \mathbf{A}_{20} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta^2 & 0 & 0 \end{pmatrix}; \mathbf{A}_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & 2\alpha & 0 \end{pmatrix};$$

$$\mathbf{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2\beta\alpha & 0 \end{pmatrix}; \quad \mathbf{A}_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}$$

Now we should write the equation for $\mathbf{B}_1$:

$$\begin{aligned} &\{p\mathbf{B}_1(-1) + (1-p)\mathbf{B}_1(1)\}A_0 - \mathbf{A}_0\mathbf{B}_1(-1) = \Lambda_1 + \mathbf{A}_1 \\ &\{(1-p)\mathbf{B}_1(-1) + p\mathbf{B}_1(1)\}A_0 - \mathbf{A}_0\mathbf{B}_1(1) = \Lambda_1 - \mathbf{A}_1 \end{aligned}$$

The invariant probability distribution for this Markov chain is $\{1/2, 1/2\}$. Thus $\Lambda_1 = 0$. Besides an averaging procedure for the function $f(y)$ leads to equation $f(-1) = -f(1)$ and the equation $\overline{B_1(y)} = 0$ is equivalent to equality $B_1(y) = yC$. Substituting this formula in the above equations we can find

$$B_1(y) = yC = y\begin{pmatrix} \frac{\beta}{2(1-p)p} & 0 & 0 \\ 0 & \frac{\beta}{2(1-p)} & 0 \\ 0 & 0 & -\frac{(1-2p)\beta}{2(1-p)p} \end{pmatrix} \tag{33}$$

Now we can move to the equation for $\mathbf{B}_1$:

$$\sum_{z=-1}^{1} \mathbf{B}_2(z)p(y,z)\mathbf{A}_0 - \mathbf{A}_0\mathbf{B}_2(y) = \Lambda_2 - y\sum_{z=-1}^{1} \mathbf{B}_1(z)p(y,z)\mathbf{A}_1 - \mathbf{A}_{20} - y\mathbf{A}_{21}$$

and derive the matrix Λ_2 by averaging of the right part of the above equation taking in the account that $\sum_{z=-1}^{1} zp(y,z) = (2p-1)y$:

$$\Lambda_2 = -\mathbf{A}_{20} - \overline{\left\{ y\sum_{z=-1}^{1} \mathbf{B}_1(z)p(y,z) \right\}}\mathbf{A}_1 = -\mathbf{A}_{20} - (2p-1)\mathbf{C}\mathbf{A}_1$$

Finily

$$\Lambda_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{(-1+2p)\beta^2}{2(1-p)} & 0 \\ -\beta^2 + \frac{(1-2p)(-1+2p)\beta^2}{(1-p)p} & 0 & 0 \end{pmatrix}$$

and the mean square Lyapunov exponents

$$S_2 := \{x \in \mathbb{R}^2 : \lim_{t\to\infty} \frac{1}{t} \ln \mathbf{E}\{|\vec{x}(t)|^2\}\}$$

are the two points set $\{\lambda_1, \lambda_2\}$ where

$$\lambda_1 = \varepsilon^2\beta^2\frac{(1-2p)}{2(1-p)} + O(\varepsilon^3), \lambda_2 = \varepsilon^2\beta^2\frac{1-3p+3p^2}{2(1-p)p} + O(\varepsilon^3)$$

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