Some New Approaches to the Construction of Attraction Domains of Periodic Motions

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Abstract − **In this paper a number of scanning algorithms for construction of basins of attraction of periodic regimes of nonlinear oscillatory systems with single degree of freedom is described. Algorithms are based on an exhaustive search of initial conditions in the given area of the phase plane and on the further analysis of the stationary motions. It should be noted that basins of attraction usually contain large homogeneous parts, belonging to the same region, whose continuous scanning does not give any useful information. This fact is taken into account here and the proposed algorithms are fulfilling the selective scanning near the boundaries of the basins, thereby forming the desired lines. As a result, the most of homogeneous regions are not tested and the number of examined starting points is significantly reduced. Here two groups of such algorithms are proposed. In the first group they are produced by repeated scanning with decreasing lead and excluding homogeneity of fragments. In the second one, the preliminary scan along the line limiting the search area is carried out and the end points of the boundaries of attraction domains are discovered. After this the selective scanning of these points is fulfilled and the demanded curves are detected gradually. These algorithms have been implemented and tested, and their comparative analysis on the example of the Duffing equation was conducted. The average time for selecting of the next initial point, the total number of points, which were used for construction and the accuracy of the obtained results were taken as the comparison criteria.**

Keywords − **Algorithm, basin of attraction, brute force, cell-mapping, scanning.**

I. INTRODUCTION

Construction of basins of attraction of periodic regimes remains one of the main problems of the analysis of dynamical systems [1]. Among the universal methods for solving of this problem one can enable approaches that are extensions of the direct Lyapunov method [2]−[4] that do not impose restrictions on the dimension of the model. In these approaches the construction of basins of the attraction is reduced to the solution of partial differential equations [2] or to the boundary value problem for the matrix Lyapunov functions [3] or to the forming of polynomial iterative procedures [4].

In simpler examples specific approaches are practiced due to the low dimensionality of the considered systems. So, for systems with one degree of freedom with use of "negative time" C. Hayashi determines the boundary lines – separatrices − that divide the phase plane into a number of areas of attraction [5]. For analysis of such systems the direct scanning of the field of given initial conditions is also often used. In technical terms, the scanning procedure, which is known in Anglophone literature as "brute force" [6], consists in choosing the points taken as the initials in the predetermined region of the phase space and the subsequent solving of the considered differential equation. But this procedure is quite resource-intensive. There are different options of its optimization known. For example, in close approaches [7], [8] belonging of the initial points to the same domain of attraction is evaluated in the stage of transition process by comparing phase coordinates of the dynamical system at moments of time, which are multiples of the period of driving forces. Generalization of such approaches is "cell-to-cell mapping" [9], in which the analysis of orbits of starting points is reduced to the analysis of corresponding sequences of cells that divide the search area into small parts, in which the phase points consistently appear at the control moments of time.

This article relates to this direction, focuses on low-dimensional systems and offers a number of computational procedures that reduce the number of scanned points by analysis of the already obtained scanned points.

II. HIGHLIGHTS OF THE PROPOSED APPROACHES

Analysis of the known results shows that, generally, the most of the analyzed areas are homogeneous and the "area" of the boundaries of basins of attraction is much smaller than the area of the plot itself. So, a question arises − whether they should be scanned or not. It also can be noted that there are no isolated areas with closed borders and usually they go outside the analyzed area to the infinity. The account of these features allows us to offer number of algorithms of point selection at scanning that expedite the receiving of basins of attraction.

The paper proposes two directions in development of such algorithms and some variants of their implementation. The concept of the first direction is that the monitoring portion is scanned multiple times with different steps. Initially, the scanning is performed with a large step and those portions that contain points corresponding to different modes are scanned with a smaller step. This procedure is repeated further until the desired accuracy, giving more detailed information. The areas that do not contain dissimilar points will not be scanned in more detail. The following describes two embodiments of this approach which differ in ways of dividing the area into sub-areas. Below they are conventionally called "field algorithms".

The second direction involves the preliminary detection of the individual points that belong to the boundary lines of basins of attraction and subsequent analysis of the nearest points. Scanning occurs along the curve that forms the outline border of the given area with the required accuracy. For this approach two alternative versions are proposed which differ in the order of forming of the mesh. These algorithms below are called "step algorithms".

III. Algorithms and Their Description

A. Forming of a Matrix Grid

In order to find basins of attraction or its cross sections for systems with one degree of freedom it is necessary to choose the search area in the form of rectangular plot on the phase plane *(x, y)*, where $y = \dot{x}$, the size of its scanning Δh as the number which characterizes the minimal distance between the points and represents the plot in the form of square grid with mesh step. The grid nodes are the points which are to be investigated. The total number of the points equals $n_x \times n_y$.

$$n_x = \left[\frac{x_{\max} - x_{\min}}{\Delta h} \right] + 1 \,, \quad n_y = \left[\frac{y_{\max} - y_{\min}}{\Delta h} \right] + 1 \,, \tag{1}$$

where $x_{\max}$, $x_{\min}$ and $y_{\max}$, $y_{\min}$ determine the size of the selected area, the sign […] denotes an integer part of a number. And, as a result, the rectangular mesh can be represented as a matrix (Fig. 1a). Only one point of the phase space corresponds to each cell of the matrix. Affiliation of the point to a certain basin of attraction is considered as the affiliation of the whole cell to the same basin too. The connection between the matrix cell and the physical values of the coordinates of the corresponding phase point is described by:

$$x_i = x_{\min} + s \cdot (i - 1) \,, \quad y_j = y_{\min} + s \cdot (j - 1) \,, \tag{2}$$

where *i* and *j* are the numbers of rows and columns of the matrix, respectively.

The use of matrix form is quite convenient as the follow-up work is carried out with the cell numbers and the scan results are obtained in the form of raster images that correspond to the graphical representation of the digital information (Fig. 1b).

Such an approach is called cell mapping [9], [10] and used in a number of sources [7], [10]–[12].

It can be mentioned, that the dimension of the matrix grid affects the resolution of the received image. If greater number of cells is used for the same size of the phase domain, more accurate image is obtained as a result.

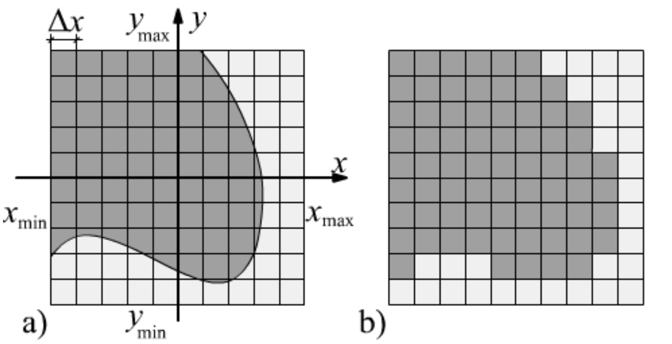


Fig. 1. Cellular representation: (a) the matrix grid of the considered site; (b) the corresponding raster image of the basin.

In description below, the first two algorithms are the “field algorithms” and two others are refered to as “step algorithms”.

B. Algorithm 1

In this case the first scanning is performed with a step, which is considerably larger than the minimum one. If the obtained points belong to different basins, the initial region is divided into subdomains and each of them is scanned and divided in a similar way until the required accuracy is reached. If the obtained points are located in the same basin, the in-depth scan is not performed. The choice of the step value, as well as of further partition, can considerably influence the results of the construction. These parameters can be defined by the user or selected automatically.

C. Algorithm 2

In comparison with the previous algorithm, this algorithm has some essential features. After preliminary rough scanning, each of four neighboring points located at the vertices of the square are compared. If these points belong to different basins, they determine the boundaries of the sub-domain for further decomposition and scanning. This scheme is illustrated in Fig. 2, where the crosses indicate points which belong to the first area and the circles − to the other one. The shaded squares denote the regions which will be scanned in more detail.

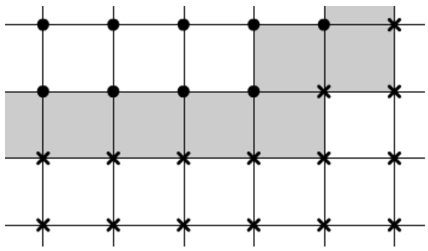


Fig. 2. The scheme of analysis of the scanned region.

The advantages of the “field scanning” are evident as the most part of the points is not subjected to the scan. But there are apparent shortcomings as during the initial scanning small fragments of the attraction areas can be missed. Fig. 3 shows the general scheme of the partition which is a tree graph. The deep traversal of the tree is realized as a recursive function. A feature of the proposed method is that many branches of this graph will be discarded during scanning.

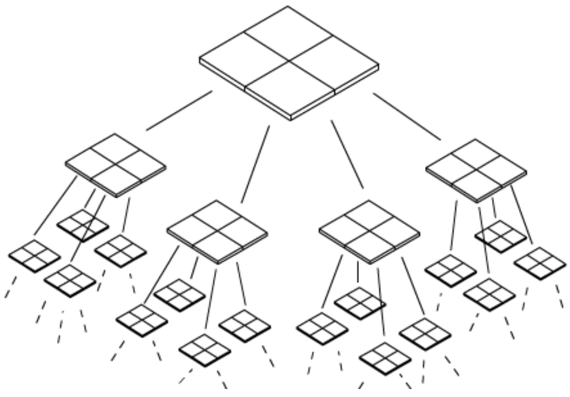


Fig. 3. A graphic representation of the partition.

D. Algorithm 3

Originally the entire perimeter of the rectangular region is being scanned with a minimal step. Every two consecutive points are compared with one another. If two such points belong to different basins, it means that the boundary of the basins passes between them. And going along the perimeter in such a way we can catch all the boundaries of the domains of attraction. Of course, this is true if the system has no isolated domains, as assumed here. These points are recorded in an array and a further scan will start with these places. Then such pairs are taken in turn and the scan is performed (Fig. 4). The third point for such pair is selected so that they form an equilateral triangle (Fig. 4a).

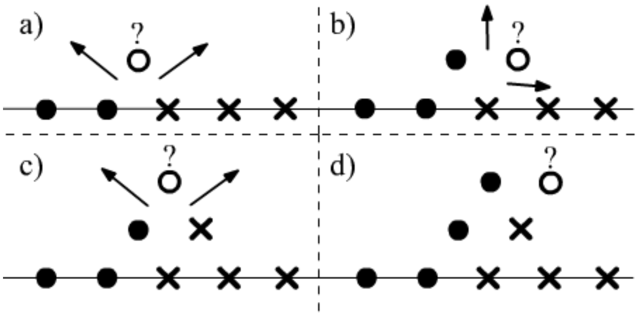


Fig. 4. Scheme: constructing the points by the 3rd algorithm.

The direction of the next choice depends on the location of the last third point (Fig. 4b). The described actions are repeated, like the previous ones (Fig. 4c) forming the contour of boundary line in the form of two rows of dots belonging to different basins (Fig. 4d).

This algorithm is not representable as a bitmap image because the vertices of a equilateral triangle cannot be bound to the matrix form. On the other hand, it is quite easy and reliable.

E. Algorithm 4

This algorithm involves the binding of the previous algorithm to the matrix. For this purpose it is necessary to select new points that form the vertices of a square (Fig. 5).

Like the previous case at the first stage of this algorithm the perimeter of the chosen rectangular region is scanned and pairs of adjacent points that belong to different basins are stored. After this, two other points are found at vertices of the square for the selected pair of points (Fig. 5a). Then verification and choice of the direction for the next step occur (Fig. 5b). The further subsequent construction of such points is performed along the desired boundary line (Fig. 5c). For such an approach the result can be represented as a matrix.

The result of the "step algorithm" is the contour of the given area and the built boundaries of basins of attraction.

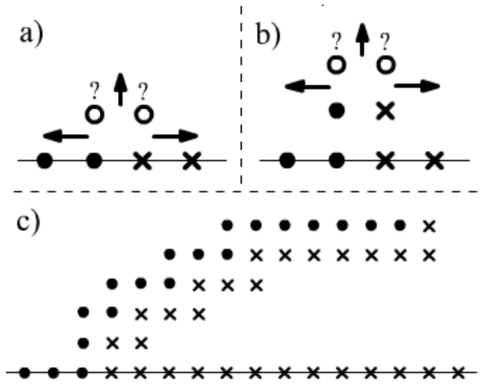


Fig. 5. Scheme: constructing of points by the 4th algorithm.

The main advantage of this algorithm is that it needs a minimal number of points for analysis, which has a positive impact on the time of construction, but unfortunately such procedure does not allow to detect isolated regions if their location is unknown.

IV. Experiments: Results and Analysis

Duffing equation was taken for the testing of the described algorithms:

$$\ddot{x} + b\dot{x} + x + \gamma x^3 = P\cos\omega t\,, \qquad (3)$$

with following parameters: $b = 0.3$, $\gamma = 0.5$, $P = 3$, $\omega = 2.3$. The scanning region was chosen as $x = (-8,8)$, $\dot{x} = (-8,8)$, initial scanning step $\Delta h = 0.04$.

Symbols and notations:

t_{solve} – the average time of analysis of a single point (determination of the belonging of the point to this or that area);

t_{self} – the average own time of algorithm for the point selection;

T_{sum} – the general working time of the program;

t_{sum} – the general analysis time of the selected points;

n – the general number of considered points, i.e. those which were selected by the algorithm;

N – the total number of points in the research region;

k – the efficiency ratio (the ratio of the considered points to its total number);

a_0, a_n, b_0, b_n – the parameters which describe the research region (here $x, \dot{x} \in (-8, 8)$).

$$t_{\text{solve}} = \frac{t_{\text{sum}}}{n}; \quad t_{\text{self}} = \frac{T_{\text{sum}} - t_{\text{sum}}}{n}; \qquad (4)$$

$$k = \frac{n}{N}100\%; \quad N = \frac{a_n - a_0}{\Delta h} \times \frac{b_n - b_0}{\Delta h}. \qquad (5)$$

So, for the chosen values of parameters the total number of points in the research region equals:

$$N = \frac{8-(-8)}{0.04} \times \frac{8-(-8)}{0.04} = 400 \times 400 = 160000\,.$$

Here are the results. The calculations were performed in *Matlab* on a computer with a processor *Intel Pentium Dual Core*, 2.2 GHz, RAM 2 GB. The results of all methods have no noticeable visual errors and are presented in Table I.

TABLE I
THE SCAN RESULTS

	Algorithm 1	Algorithm 2	Algorithm 3	Algorithm 4
T_{sum}	6084	1956	1307	1207
t_{sum}	6073	1897	1299	1200
n	38989	11950	8172	7514
t_{solve} (s)	0.15576	0.16317	0.15895	0.15970
T_{self} (s)	0.00035	0.00050	0.00097	0.00093
k (%)	24	7.4	5.1	4.7

Algorithm 1 shows the highest speed of point selection and the longest runtime. It analyzed 24 % of total number of points, which is the highest among all algorithms. Consideration of a large number of points ensures that this method has a high degree of reliability and you can expect small errors when it is used. The result of this algorithm is shown in Fig. 6.

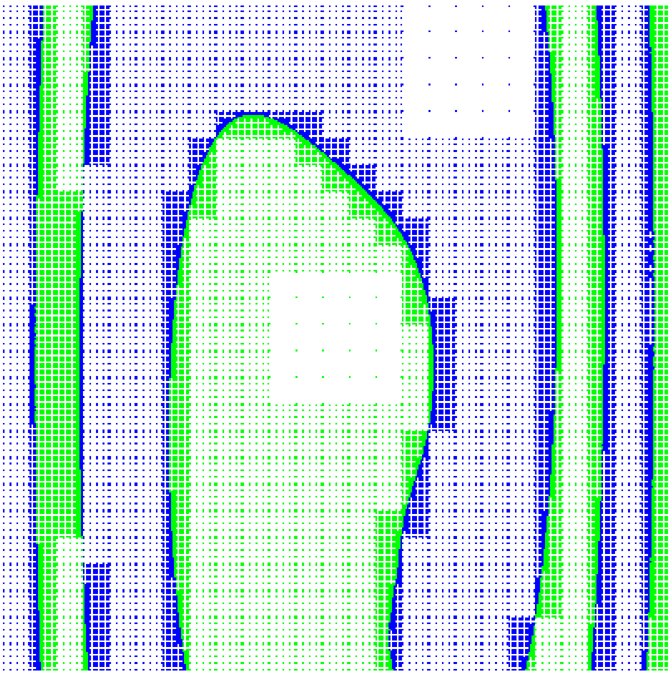


Fig. 6. Points that have been used by Algorithm 1.

Algorithm 2 requires far less points. It is three times faster than the previous one and its own time for selection of new point is only a little longer than before. The average time of solving the differential equation also increases. Perhaps this is due to the fact that the algorithm determines more points near the boundary for analysis of which it requires more calculations. This method is faster, but is not reliable and can give a large percentage of errors. Its accuracy depends upon the program settings and the form of the domains. The results of this method are demonstrated in Fig. 7.

Algorithms 3 and 4 are more promising than the previous ones, since they require fewer numbers of points. The result of Algorithm 4 is depicted in Fig. 8. It shows that all tested points participate in the formation of the boundary. The result of

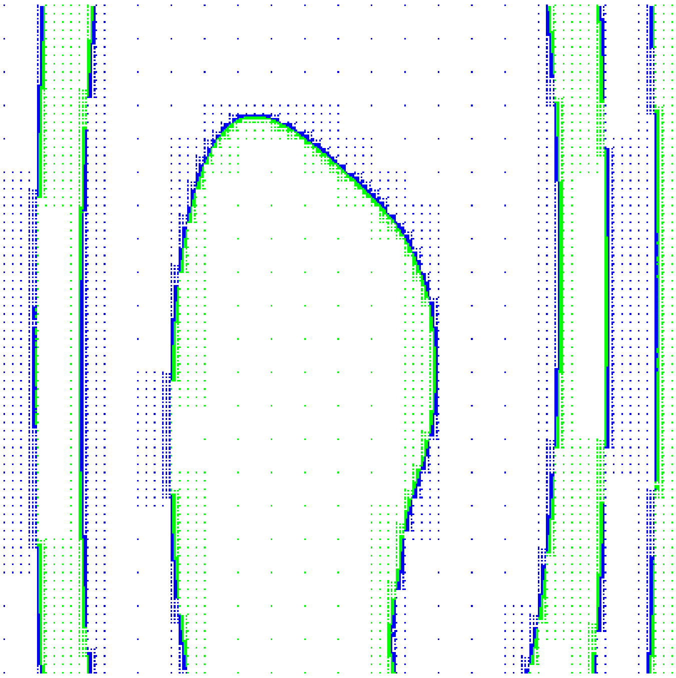


Fig. 7. Points that have been used by Algorithm 2.

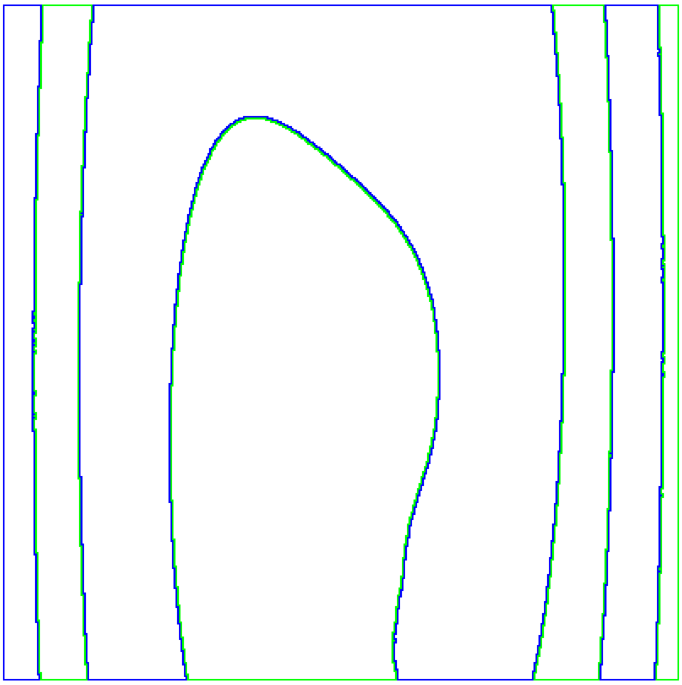


Fig. 8. The result of the algorithm 4.

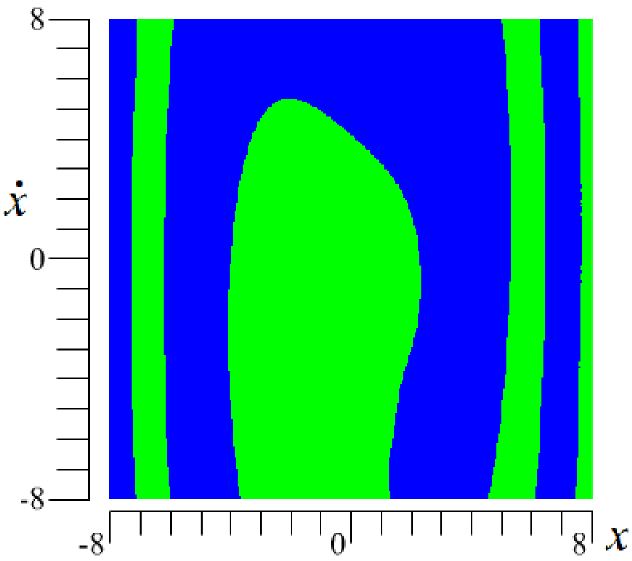


Fig. 9. Final image.

Algorithm 3 cannot be represented as a raster image but the received result practically coincides with Fig. 8. The own selection time of both algorithms is higher than that of the first two, but it practically does not affect the overall execution time. Fig. 9 shows the final image of the basins in the given area of the phase plane.

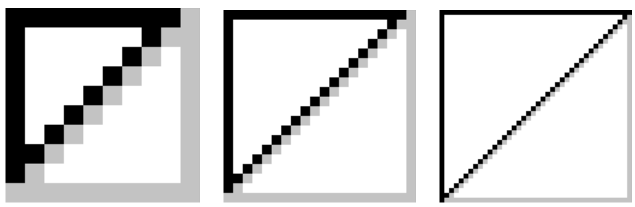
Efficiency ratio is a relative characteristic that depends not only on the algorithm, but also on the shape of searched basins and scanning step. This fact is illustrated by Table II, where the scanning results of the certain plot with different steps are presented. Calculations were performed by Algorithm 4, as it has the shortest operating time.

TABLE II

IMPACT OF THE SCANNING STEP ON THE EFFICIENCY RATIO

	Δh = 0.08	Δh = 0.04	Δh = 0.02
N	40000	160000	640000
n	3767	7514	16047
k	9.4	4.7	2.5

Analysis of the obtained results shows a certain proportional dependence between parameters *k* and Δh, i.e. with step reduction the efficiency ratio becomes smaller. This phenomenon is illustrated in Fig. 10 and can be based theoretically.



Δh = 16 Δh = 8 Δh = 4

Fig. 10. “Boundary lines” for different values of scanning steps.

Consider a square plot and assume that its perimeter is the boundary line of the desired basin. After analysis of this section a series of points located on the perimeter is obtained. Then, after scanning with a smaller step, we will get a thiner line in the final image. Connection between the scanning step and efficiency ratio can be described by the following: *S* is the total area of the square, *P* – the area of the boundary strip, *L* – the side length of the square, *k* – the efficiency ratio, Δh – the scanning step or width of the strip.

Then:

$$P = (L - \Delta h)\cdot 4\Delta h, \tag{6}$$

$$S = L^2, \tag{7}$$

$$k_e = \frac{P}{S} = \frac{(L-\Delta h)\cdot 4\Delta h}{L^2}, \tag{8}$$

$$L = const,\ L \gg \Delta h,\ \Rightarrow k_e \sim \Delta h. \tag{9}$$

Of course this result is correct quantitatively only for this concrete example, but qualitatively this tendency is general for domains of any other forms.

V. Conclusion

All presented methods help to accelerate the process of obtaining the domains of attraction and to provide high accuracy. Each of them has certain advantages and can be further developed. In particular, additional checking can be added to the “field algorithms” if small fragments of the basins are missed and the additional search of isolated basins can be entered into the “step algorithms”.

At the same time, it should be noted, that due to the exponential growth of the amount of computations with the increase of dimensions, the use of scanning techniques for the analysis of dynamical systems of higher dimensions is very problematic. Although certain generalizations for three dimensional space, in particular, are quite possible. Yet, the use of scanning methods for finding low dimensional cross sections of attraction domains seems to be more a realistic option of their application in multivariate cases.

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Aleksandrs Smirnovs, Valērijs Belovodskis. Jauna pieeja periodisku kustību pievilkšanas apgabalu noteikšanā
Darbā aprakstīti vairāki skenējošie algoritmi periodisku režīmu pievilkšanas apgabalu noteikšanai nelineārām sistēmām ar vienu brīvības pakāpi. Algoritmi pamatojas uz fāzes plaknes izvēlētās daļas sākuma noteikumu mainīšanu un sekojošo kustību analīzi. Jāatzīmē, ka pievilkšanas apgabali parasti satur plašas viendabīgas daļas, kuru skenēšana nesniedz nekādu noderīgu informāciju. Tādēļ dažos piedāvātajos algoritmos atšķirībā no nepārtrauktās skenēšanas lieto izlases skenēšanu gar meklējamo pievilkšanas robežu. Šajā procesā veidojas meklētās līnijas. Rezultātā liela daļa viendabīgo apgabalu netiek testēti, un pārbaudāmo punktu skaits būtiski samazinās. Darbā aprakstītas divas algoritmu grupas. Pirmajā grupā pētījumu zona tiek atkārtoti skenēta ar samazinātu soli, neņemot vērā fragmentu viendabību. Otrajā grupā vispirms notiek iepriekšēja skenēšana gar pētījumu apgabala robežlīniju, nosaka uz tās atrodošos pievilkšanas apgabalu galapunktus, un tam seko izlases skenēšana gar pakāpeniski veidotajām meklētajām līknēm. Ieteiktie algoritmi realizēti ar programmām, veikta to testēšana un salīdzinošā analīze, izmantojot Dufinga vienādojumu. Izvēlēti šādi salīdzināšanas kritēriji: kārtējā sākumpunkta izvēles vidējais laiks, kopējais punktu skaits, kurus izmanto apgabalu noteikšanā, un iegūto rezultātu precizitāte.