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A STOCHASTIC MODEL OF THE LOGISTIC SYSTEM DYNAMICS

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Abstract. The stochastic vehicle allocation problem addresses the movement of vehicles between locations over a given planning horizon. The demand for vehicles to carry loads between locations is uncertain, and vehicles are assumed able to handle several loads over the course of the planning horizon. Assuming random and coming at random time moments demands, we construct a stochastic model for this transport logistic scheme and derive Gaussian approximation for transport and stock level of goods dynamics. The proposed stochastic model in tandem with stochastic approximation procedure permits to take into account random character of demand for freight services and to supplement the classical deterministic analysis with Gaussian approximation for possible random deviations.

Keywords: stochastic approximation, transport allocation problem, Gaussian approximation

Mathematics Subject Classification: Primary 92B05, 60H30; Secondary 60G57, 37H10

1 Introduction

The beginning of the twenty-first century and the last decades in the twentieth century have indeed been an exciting era for research and development in traffic and transportation systems. Considerable advances in real-time traffic detection, data processing, communications, and control methods have enabled new frontiers in both developing a deeper understanding of the nature of traffic and transportation as well as opening up new ways to manage and control the transportation system in response to the actual conditions. (see [1,5,6,7,8] and references there).
Even for most simple logistic dynamical system consisting of a wholesale store of capacity A, a retail store of capacity R and automobiles which are taking part in goods delivery from a wholesale store to a retail store the author of paper [1] by means of imitation modelling succeeded in finding such a complex mode of the operation as limit cycles and other irregular attractors. But in reality any transport logistics model is dependent at random demand and operates at random environment. Besides, a time moment for restocking of goods also is a random value. This means that for quantitative analysis for goods growth we have to calculate

not only given by deterministic dynamical system stock level of goods bet also to estimate possible random deviations on these idealized representations. To do this in our paper we consider some complicate proposed in [1] deterministic model assuming that the demand be random and coming at random time moments. The expressed in paper [1] mathematical model for dynamical analysis of the above transport logistics scheme is system of three dimensional ordinary differential equations.

$$\begin{cases} \frac{d}{dt}x(t) = kz(t)(R - y(t)), \\ \frac{d}{dt}y(t) = a\,\mathrm{R}^{-1}x(t)(\mathrm{R} - y\ (t)) - \varepsilon\int_U B(u)y\ (t)\mu\ (du, dt), \\ \frac{d}{dt}z(t) = -hR^{-1}x(t)(R - y(t)) + cA^{-1}(A - z(t)). \end{cases} \tag{1}$$

with right part, that dependent on the number of involving in goods delivery transport $x(t)$ and stock levels of goods $y(t)$ and $z(t)$ in the corresponding stores. This model constructed under assumption that, for any $t \geq 0$:

- the increments $\Delta x(t) := x(t+\Delta) - x(t)$ of involving in goods delivery number of trucks are proportional to Δ multiplied by to stock $z(t)$ at wholesale store and a number of vacancies $\mathrm{R} - y(t)$ at retail store:
  $\Delta x(t) := x(t+\Delta) - x(t) = kz(t)(R - y(t))\Delta$;
- the increments $\Delta y(t) := y(t+\Delta) - y(t)$ of stock levels of goods are proportional to Δ multiplied by involving in goods delivery number of trucks $x(t)$, a number of vacancies $\mathrm{R} - y(t)$ at retail store, after deduction of ordering for goods $by(t)\Delta$;
  $\Delta y(t) := y(t+\Delta) - y(t) = bx(t-\Delta)(1 - \mathrm{R}^{-1}y(t-\Delta)) - h(\xi(t-\Delta))y(t-\Delta)$;
- the increments $\Delta z(t) := z(t+\Delta) - z(t)$ of stock levels of goods are proportional to Δ multiplied by $x(t)$, a number of vacancies $\mathrm{A} - z(t)$ at wholesale store, after deduction of goods transportable from wholesale store to retail store:
  $\Delta z(t) := z(t+\Delta) - z(t) = c(1 - \mathbf{A}^{-1}z(t-\Delta)) - hx(t-\Delta)(1 - \mathrm{R}^{-1}y(t-\Delta))$.

To take into account random properties of demand for goods we have to model a demand at the time interval $[t, t+\Delta)$ as a random variable that can arrive or not with dependent on interval length probability. That is why we propose for dynamical analysis of the above logistic transportation scheme a stochastic model given by following finite-difference approximation:

$$x_\varepsilon(t_{k+1}) = x_\varepsilon(t_k) + kz_\varepsilon(t_k)(R - y_\varepsilon(t_k))\Delta,$$

$$y(t_{k+1}) = y(t_k) + \Delta_y,$$

$$\Delta_y = \begin{cases} a\mathrm{R}^{-1} x_\varepsilon(t)(t)(\mathrm{R} - y_\varepsilon(t)(t))\Delta - \varepsilon B(\xi_k) y(t_k), & \text{if } \Delta \ge \eta_k, \\ a\mathrm{R}^{-1} x_\varepsilon(t)(t)(\mathrm{R} - y_\varepsilon(t)(t))\Delta, & \text{if } \Delta < \eta_k; \end{cases} \tag{2}$$

$$\{\xi_k, k \in N\} \sim i.i.d.\ \mathrm{R}(0,1), \{\eta_k, k \in N\} \sim i.i.d.\ P(\eta_k \ge t) = \varepsilon^{-1} e^{-\varepsilon^{-1} t};$$

$$z_\varepsilon(t_{k+1}) = z_\varepsilon(t_k) - hR^{-1} x_\varepsilon(t_k)(R - y_\varepsilon(t_k))\Delta + cA^{-1}(A - z_\varepsilon(t_k)),$$

where $t_k = k\Delta, k \in N$, ε is a small positive parameter, and $\Delta_y(t_k)$ is a random sequence defined by dependent on two identically independent distributed (i.i.d.) independent uniform R(0,1) distributed series $\{\xi_k, k \in N\}$ and exponentially distributed with parameter $\varepsilon^{-1}\Delta$ series $\{\eta_k, k \in N\}$ as follows:

$$\begin{aligned} &P(\Delta_y(t_k) = 0 / \Delta < \eta_k) = 1 - e^{-\varepsilon^{-1}\Delta}, \\ &P(\Delta_y(t_k) = \varepsilon b(u) y_\varepsilon(t_k) / \Delta \ge \eta_k, \xi_k = u) = e^{-\varepsilon^{-1}\Delta}, \\ &k \in N, u \in [0,1]. \end{aligned} \tag{3}$$

This means that there are random time moments $\{\tau_k, k \in N\}$ when the trajectory for stock levels of goods $y_\varepsilon(t)$ has small jumps $\varepsilon b(\xi_k) y_\varepsilon(t_k)$ bet these jumps occur very close: $\forall k \in \mathrm{N} : \mathrm{E}\{\tau_{k+1} - \tau_k\} = \varepsilon$. The sample trajectories for equations (2) - (3) with parameters $\varepsilon = 0.01, \Delta = 0.001, k = 1, h = 1, R = 10, \mathrm{c} = 1,\ A = 100, b(u) = 2bu, b = 0.25$ and trajectories for solutions of equitation (1) with initial conditions $x_\varepsilon(0) = x(0) = 2, y_\varepsilon(0) = y(0) = 2, z_\varepsilon(0) = z(0) = 2$ are shown at the Fig.1. As we can see most dependent on random demand are dynamics for stock levels of goods $y_\varepsilon(t)$.

At the next sections applying the stochastic averaging method [3] we derive approximative solution for (2) -(3) as a three dimensional Gaussian process and discuss a behaviour of mean value and variances for stock levels of goods $\{z_\varepsilon(t), y_\varepsilon(t), z_\varepsilon(t)\}$.

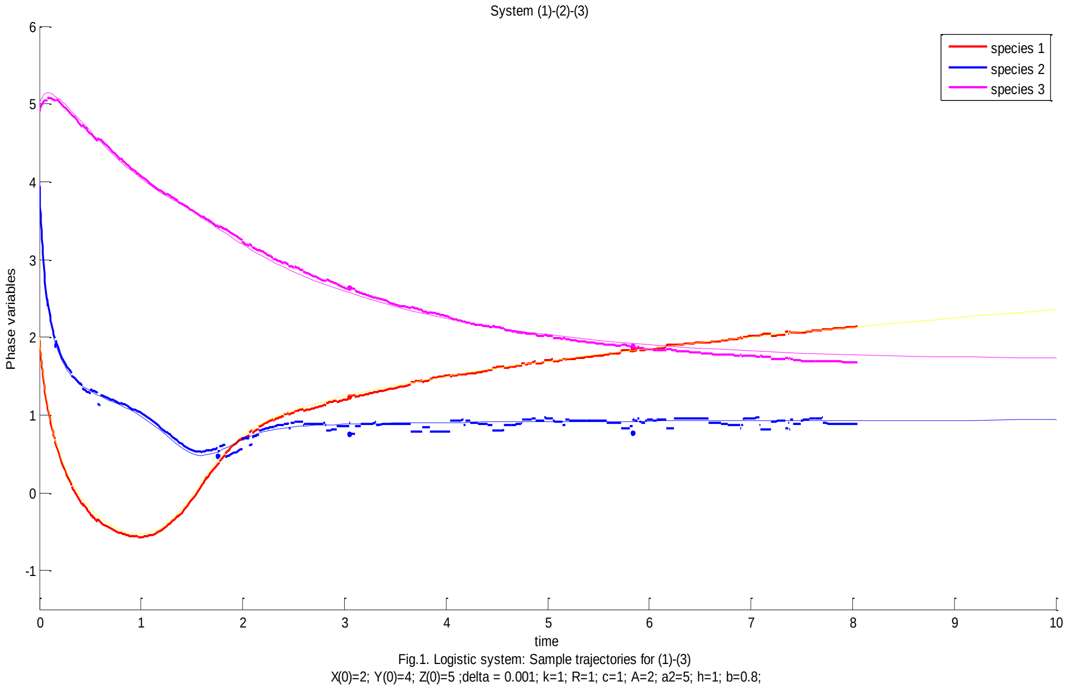

Fig. 1. Sample trajectory for (1) (unbroken line) and corresponding solutions of (2) - (3) (broken lines).

2 Diffusion approximation procedure

The defined in previous section stochastic dynamical system in more general form has been analysed in our previous paper [2]. The corresponding to finite-difference equation (2) - (3) random process possess Markov property and may be analysed through intermediary of generator [3]

$$L(\varepsilon)v(x,y,z) := \lim_{\Delta\to 0} \mathrm{E}\{v(x_\varepsilon(t+\Delta)-x_\varepsilon(t), y_\varepsilon(t+\Delta)-y_\varepsilon(t), z_\varepsilon(t+\Delta)-z_\varepsilon(t) / x_\varepsilon(t)=x, y_\varepsilon(t)=y, z_\varepsilon(t)=z\} =$$
$$=\frac{1}{\varepsilon}\int_U [v(x+\varepsilon g_x(x,y,u), y+\varepsilon g_y(x,y,u), z+\varepsilon g_z(x,y,z,u)) - v(x,y,z)]\pi(du);$$

$$\mathcal{L}(\varepsilon)v(x,y,z) := \lim_{\Delta\to 0}\frac{1}{\Delta}\mathrm{E}\{v(x_\varepsilon(t_k+\Delta), y_\varepsilon(t_k+\Delta), z_\varepsilon(t_k+\Delta)) - v(x,y,z)|_{x_\varepsilon(t_k)=x, y_\varepsilon(t_k)=y, z_\varepsilon(t_k)=z}\} =$$
$$=\left\{kz(\mathrm{R}-y)\frac{\partial}{\partial x} + a\,\mathrm{R}^{-1}x(\mathrm{R}-y)\frac{\partial}{\partial y} + [c\mathrm{A}^{-1}(\mathrm{A}-z) - h\,\mathrm{R}^{-1}x(\mathrm{R}-y)]\frac{\partial}{\partial z}\right\}v(x,y,z) + \tag{4}$$
$$+\frac{1}{\varepsilon}\int_0^1 [v(x, y+\varepsilon b(u)y, z) - v(x,y,z)]du,$$

where $v(x,y,z)$ is an arbitrary sufficiently smooth bounded function . Now we have to derive a limit $\lim_{\varepsilon\to 0}\mathcal{L}(\varepsilon)v(x,y,z) := \mathcal{L}v(x,y,z)$, where

$$\mathcal{L} = kz(\mathrm{R}-y)\frac{\partial}{\partial x} + \left[a\,\mathrm{R}^{-1}x(\mathrm{R}-y) - by\right]\frac{\partial}{\partial y} + [c\mathrm{A}^{-1}(\mathrm{A}-z) - h\,\mathrm{R}^{-1}x(\mathrm{R}-y)]\frac{\partial}{\partial z} \tag{5}$$

and $b = \int_0^1 b(u)du$. The operator (5) can be interpreted as an infinitesimal operator [9] for defined by system (1) continuous semigroup and therefore [2] for sufficiently small $\varepsilon > 0$ sample trajectories of defined by finite-difference equation (2) - (3) random dynamical system we can approximate by corresponding solutions of equation (1), that is, if $x_\varepsilon(0) = x(0), y_\varepsilon(0) = y(0), z_\varepsilon(0) = z(0),$ then for any $T > 0$

$$\mathrm{P}\left\{\lim_{\varepsilon\to 0}\sup_{0\le t\le T}[|\,x_\varepsilon(t) - x(t)\,| + |\,y_\varepsilon(t) - y(t)\,| + |\,z_\varepsilon(t) - z(t)\,|] = 0\right\} = 1 \tag{6}$$

As it has been proven in [2] the deviations of solutions (2) - (3) on corresponding solutions of (1) have an order $\sqrt{\varepsilon}$ and we may analyse these deviations applying diffusion approximation procedure to no homogeneous three dimensional Markov process

$$X_\varepsilon(t) = \frac{x_\varepsilon(t) - x(t)}{\sqrt{\varepsilon}}, Y_\varepsilon(t) = \frac{y_\varepsilon(t) - y(t)}{\sqrt{\varepsilon}}, Z_\varepsilon(t) = \frac{z_\varepsilon(t) - z(t)}{\sqrt{\varepsilon}} \tag{7}$$

with zero initial conditions. The same as before we should derive a generator for (7)

$$\Lambda(\varepsilon)v(X,Y,Z) := \lim_{\Delta\to 0}\frac{1}{\Delta}\mathrm{E}\{v(X_\varepsilon(t_k+\Delta), Y_\varepsilon(t_k+\Delta), Z_\varepsilon(t_k+\Delta)) - v(X,Y,Z)\big|_{X_\varepsilon(t_k)=X, Y_\varepsilon(t_k)=Y, Z_\varepsilon(t_k)=Z}\}$$

and pass to limit $\lim_{\varepsilon\to 0}\Lambda(\varepsilon)v(X,Y,Z) := \Lambda v(\mathrm{X},Y,Z)$, where

$$\begin{aligned}\Lambda v(\mathrm{X},Y,Z) = \Big\{&[-kz(t)Y + (\mathrm{R}-y(t))Z]\frac{\partial}{\partial X} + \\ &+[-a\,\mathrm{R}^{-1}x(t)Y + a\,\mathrm{R}^{-1}X(\mathrm{R}-y(t))]\frac{\partial}{\partial Y} + \frac{1}{2}\beta^2 y^2(t)\frac{d^2}{dY^2} + \\ &+[-c\mathrm{A}^{-1}\mathrm{Z} + h\,\mathrm{R}^{-1}x(t)Y - X(\mathrm{R}-y(t))]\frac{\partial}{\partial Z}\Big\}v(\mathrm{X},Y,Z)\end{aligned} \tag{8}$$

and $\beta^2 = \int_0^1 b^2(u)du$. This operator can be interpreted [3] as a generator for no homogeneous Markov process $\{X(t), Y(t), Z(t), t \ge 0\}$ which satisfies to the systems of two ordinary equations and one stochastic Ito equation:

$$\frac{d}{dt}X(t) = -kz(t)Y(t) + (\mathrm{R}-y(t))Z(t), \tag{9}$$

$$dY(t) = [-a\,\mathrm{R}^{-1}x(t)Y(t) + a\,\mathrm{R}^{-1}X(t)(\mathrm{R}-y(t))]dt + \beta y(t)dw(t), \tag{10}$$

$$\frac{d}{dt}Z(t) = -c\mathrm{A}^{-1}Z(t) + h\,\mathrm{R}^{-1}x(t)Y(t) - X(t)(\mathrm{R}-y(t)), \tag{11}$$

with initial conditions $\{X(0) = 0, Y(0) = 0, Z(0) = 0\}$. As it has been proved in [2] finite dimensional distributions of the defined by equations (2) - (3) Markov process $\{x_\varepsilon(t), y_\varepsilon(t), z_\varepsilon(t)\}$ may be approximated by corresponding finite dimensional distributions of the process

$$\bar{x}_\varepsilon(t) = x(t) + \sqrt{\varepsilon}X(t),\ \bar{y}_\varepsilon(t) = y(t) + \sqrt{\varepsilon}Y(t),\ \bar{z}_\varepsilon(t) = z(t) + \sqrt{\varepsilon}Z(t). \tag{12}$$

Unfortunately, we cannot analyse variance separately approximation for stock levels of goods given by equation (10). We have to derive and solve the system of differential equations for all elements of a covariance matrix for the three dimensional Gaussian random vector $\{X(t), Y(t), Z(t)\}$:

$$
\begin{aligned}
&\frac{d}{dt} q_{XX}(t) = -2kz(t) q_{XY}(t) + (\mathrm{R} - y(t)) q_{XZ}(t), \\
&\frac{d}{dt} q_{XY}(t) = -kz(t) q_{YY}(t) + (\mathrm{R} - y(t)) q_{YZ}(t) - a\,\mathrm{R}^{-1} x(t) q_{XY}(t) + a\,\mathrm{R}^{-1} q_{XX}(t)(\mathrm{R} - y(t)), \\
&\frac{d}{dt} q_{XZ}(t) = -kz(t) q_{YZ}(t) + (\mathrm{R} - y(t)) q_{ZZ}(t) - c\mathrm{A}^{-1} q_{XZ}(t) + h\,\mathrm{R}^{-1} x(t) q_{XY}(t) - q_{XX}(t)(\mathrm{R} - y(t)), \\
&\frac{d}{dt} q_{YY}(t) = -2a\,\mathrm{R}^{-1} x(t) q_{YY}(t) + 2a\,\mathrm{R}^{-1} q_{XY}(t)(\mathrm{R} - y(t)) + \beta^2 y^2(t), \\
&\frac{d}{dt} q_{YZ}(t) = -(a\,\mathrm{R}^{-1} + c\mathrm{A}^{-1}) x(t) q_{YZ}(t) + a\,\mathrm{R}^{-1} q_{XZ}(t)(\mathrm{R} - y(t)) + h\,\mathrm{R}^{-1} x(t) q_{YY}(t) - q_{XY}(t)(\mathrm{R} - y(t)), \\
&\frac{d}{dt} q_{ZZ}(t) = -2c\mathrm{A}^{-1} q_{ZZ}(t) + 2h\,\mathrm{R}^{-1} x(t) q_{YZ}(t) - 2q_{XZ}(t)(\mathrm{R} - y(t)),
\end{aligned} \tag{13}
$$

with zero initial conditions. Applying the Runge-Kutta method for solution of equations (13) we can calculate approximation for covariance matrix for stock levels of goods for with the same parameters in (1)-(2)-(3) and initial values as for the Fig.1.

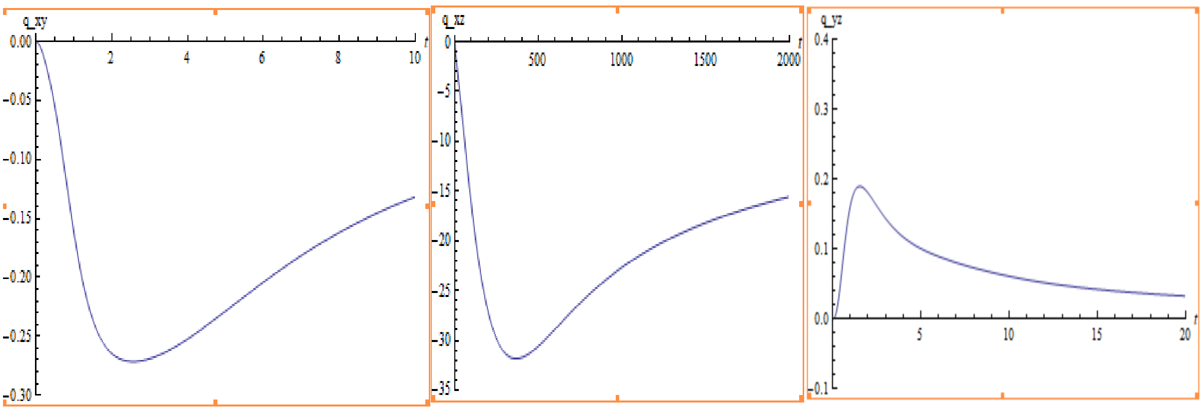

Fig. 2. Covariances $q_{XY}(t), q_{XZ}(t), q_{YZ}(t)$.

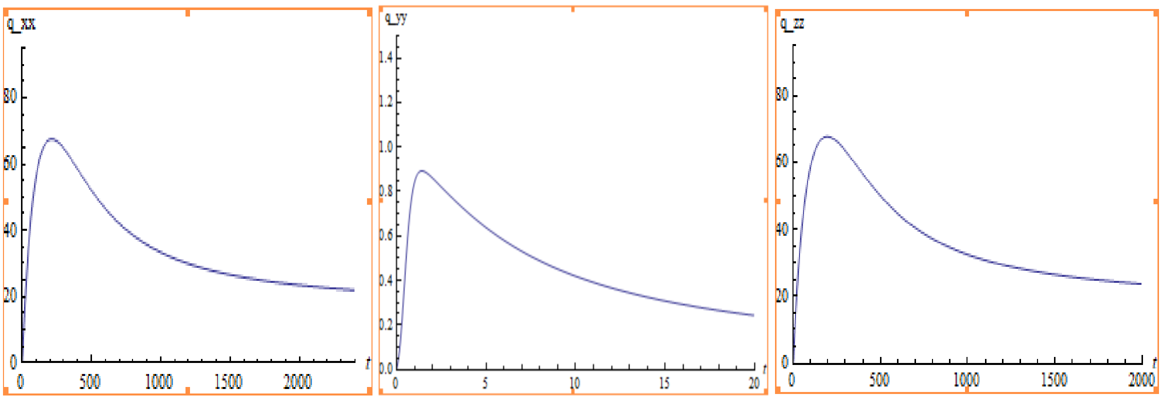

Fig. 3. Variances $q_{XX}(t), q_{YY}(t), q_{ZZ}(t)$.

3 Conclusion

The proposed stochastic model for transport logistics in a form of nonlinear difference-differential equations with stochastic Poisson type increments in tandem with stochastic approximation procedure permits to take into account random character of demand for freight services and to supplement the classical deterministic analysis with Gaussian approximation for possible random deviations.

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