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DEDUCTIVE ENUNCIATION OF KINEMATICS OF THE CLASSICAL MECHANICS: THE SECOND DERIVATIVE OF THE VECTOR IN ONE AND TWO SYSTEMS OF COORDINATES ROTATING FROM EACH OTHER

KLASISKAS MEHĀNIKAS KINEMĀTIKAS DEDUKTĪVS IZKLĀSTS: OTRAIS VEKTORA ATVASINĀJUMS VIENĀ UN DIVAS KOORDINĀTAS SISTĒMAS, KURAS GRIEŽAS VIENS RELATĪVI OTRA

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1. Introduction

The basic metric sizes (length and angle) [1] should be variables and have derivatives (in an ideal, infinite number, starting with the first derivative) for logically clear and correct transition from geometry and vector algebra to differential geometry and the vector analysis and further to kinematics.

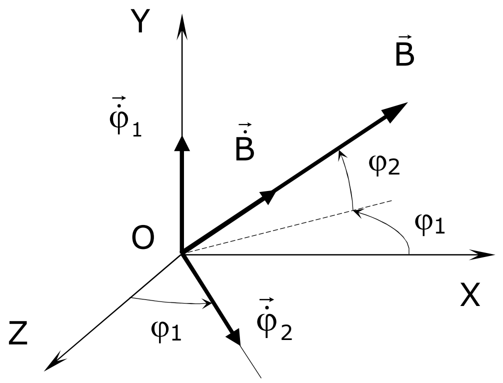

Fig. 1.

If to designate: $\vec{B}$ – an any vector; B – length of this vector; φ_1 and φ_2 – the angles determining a direction of this vector concerning considered system of coordinates (Fig. 1); $\dot{\vec{B}}, \ddot{\vec{B}}, \dddot{\vec{B}}$ and so on, $\dot{B}, \ddot{B}, \dddot{B}$ etc., $\dot{\varphi}_1, \ddot{\varphi}_1, \dddot{\varphi}_1$ etc. $\dot{\varphi}_2, \ddot{\varphi}_2, \dddot{\varphi}_2$ etc. – derivatives on time of the mentioned above variables the vector $\dot{\vec{B}}$ of the first derivative of a considered vector should necessarily somehow include derivatives $\dot{B}, \dot{\varphi}_1, \dot{\varphi}_2$, the vector $\ddot{\vec{B}}$ of the second derivative of a considered vector should necessarily somehow include derivatives $\ddot{B}, \ddot{\varphi}_1, \ddot{\varphi}_2$ and so on. It allows to proceed instead of used usually constructive definitions of derivatives of a vector (the first and higher orders), to the descriptive definitions uniting linear and angular speeds (derivatives of lengths and angles) and further linear and angular acceleration which all together and are objects of consideration in kinematics of a point, and then and a rigid body.

It is possible to word the beginnings of the vector analysis, differential geometry and kinematics in this case under the following plan:

1. Consideration of one variable vector in one system of coordinates (SC), including vectors of the first and more the senior derivatives of a considered vector.
2. Consideration of one variable vector and vectors of its derivatives in two SC, rotating with the

given angular speed from each other, and with the conterminous beginnings of coordinates (or with constant shift).
3. Consideration of one variable vector and vectors of its derivatives in two SC, rotating with the given angular speed from each other, and with the beginnings of coordinates moving from each other.

If a considered vector to count a radius-vector of a point after consideration of first two items (it is wording in [2]) it is possible to start to wording kinematics of a point and to study kinematics of a rigid body with the fixed point. By consideration of the third item it is possible to enter concept of the connected vector along with free and sliding vectors (angular speed) and then to wording so-called "complex" movements of a point and a rigid body.

2. The knowledge about the first derivative of a vector

Let's remind briefly some basic results of work [2] without which the further wording is impossible (in view of some new designations and formulas).

The formula of the first derivative of a vector (and any subsequent if the previous derivative to consider simply a vector) to deduce as the expression including a derivative $\vec{\dot{B}}$ of length of a vector $\vec{B}$ and derivative $\vec{\dot{\varphi}}_1, \vec{\dot{\varphi}}_2$ two angles φ_1 and φ_2, determining direction of this vector concerning considered system of coordinates (see Fig. 1), as follows:

$$\dot{\vec{B}} = \vec{\dot{B}} + (\vec{\dot{\varphi}}_1 + \vec{\dot{\varphi}}_2) \times \vec{B}, \tag{1}$$

where
$$\vec{\dot{B}} \equiv \dot{B}\ \vec{e}_B\,. \tag{2}$$

Here $\dot{B} \equiv dB/dt$ – a derivative on time of length of a vector $\vec{B}$, a scalar as $B \equiv |\vec{B}| \equiv \vec{B}/\vec{e}_B$ – the length (module) of a vector $\vec{B}$ is a scalar, and $\vec{e}_B$ – an unit vector of straight lines, collinear to a vector $\vec{B}$ (dimensionless, dimension have variables $B, \dot{B}, \ddot{B}$ etc.; analogously to angular speed, its components and derivatives).

The vector $\vec{\dot{B}}$ is directed along a vector $\vec{B}$ if the length of this vector is increased and on the contrary if its length decreases.

If rotation of a considered vector $\vec{B}$ concerning considered SC is given by a vector of angular speed (theoretically, practically this vector to define it is impossible)

$$\vec{\omega}_B \equiv \omega_B\ \vec{e}_o\,, \tag{3}$$

where $\vec{e}_o$ - an unit vector of a straight line around of which rotates considered a vector $\vec{B}$, and ω_B – length of a vector of angular speed,

that the formula (1) gets a kind:

$$\dot{\vec{B}} = \vec{\dot{B}} + \vec{\omega}_B \times \vec{B}. \tag{4}$$

Formulas (1) and (4) are two kinds of one formula of the first derivative of the vector giving an opportunity of its descriptive definition (see [2, p.100]).

The vector of angular speed $\vec{\omega}_B$ of a vector $\vec{B}$ can be to decompose in considered SC on three components by two ways:

$$\vec{\omega}_B = \vec{\dot{\varphi}}_1 + \vec{\dot{\varphi}}_2 + \vec{\dot{\varphi}}_3 = \vec{\omega}_{Bx} + \vec{\omega}_{By} + \vec{\omega}_{Bz}\,, \tag{5}$$

where φ_3 – an angle of turn around of a straight line conterminous to a vector $\vec{B}$ (if the vector $\vec{\omega}_B$ is not given then the magnitude of this angle and a direction of turn can be anyone);
$\vec{\omega}_{Bx} = \left(\vec{\omega}_B \cdot \vec{e}_x\right)\vec{e}_x$ – a component of a vector of angular speed along an axis x
(and so on for other axes of coordinates).

At the decision of practical problems it is necessary to consider frequently two and more SC, rotating from each other. Derivatives of each considered vector can be determined concerning each of these SC by formulas (1) or (4) (see [2, p.104]), and then the formula of distinction or expression of connection of these derivatives can be deduced by subtraction of one of the received expressions from another (for example, for SC $OX_1Y_1Z_1$ and $OX_2Y_2Z_2$, rotating from each other with angular speeds $\vec{\omega}_2^{(1} = -\vec{\omega}_1^{(2}$) as

$$\vec{\dot{B}}^{(1} = \vec{\dot{B}}^{(2} + \vec{\omega}_2^{(1} \times \vec{B} \text{ or } \vec{\dot{B}}^{(2} = \vec{\dot{B}}^{(1} + \vec{\omega}_1^{(2} \times \vec{B}\,. \tag{6}$$

We use the top index as cipher with a bracket equally for a designation as rotations, and differentiation concerning considered SC (the bracket is used that there was a difference from exponentiation).

Let's note once again, that this formula shows, than vectors of the first derivative of a vector in two rotating from each other SC differ, but it stay not clear at a deduction of this formula in other ways what in these formulas are equally.

3. The second derivative of a vector in one considered system of coordinates

Based upon above wording, our following purpose is the deduction of the formula for a vector $\ddot{\vec{B}}$ – the second derivative of a considered vector in the considered system of coordinates including vectors $\vec{\ddot{B}}, \vec{\ddot{\varphi}}_1, \vec{\ddot{\varphi}}_2$. Such formula can be received from expression of influence on a vector of operators of infinitesimal expansion of space (or shift of a point of the end of a vector) and infinitesimal torsion (see [2, pp.100-102]), considering infinitesimal quantity of the second order, but it is easier for receiving, differentiating the formulas (1) or (4) deduced in the same work (not forgetting at the same time that all vectors should be differentiated by the same formulas).

Differentiating the formula (1), we shall receive (if we use firstly symbols of differentiation by Leibniz (1646-1716) which more bulky but are more clear in comparison with symbols of differentiation over Newton (1642-1727))

$$\frac{d^2}{dt^2}\left(\vec{B}\right) = \frac{d}{dt}\left(\frac{d}{dt}\vec{B}\right) = \frac{d}{dt}\left(\overrightarrow{\frac{d}{dt}|\vec{B}|}\right) + \left(\frac{d}{dt}\left(\overrightarrow{\frac{d\varphi_1}{dt}}\right) + \frac{d}{dt}\left(\overrightarrow{\frac{d\varphi_2}{dt}}\right)\right) \times \vec{B} + \left(\overrightarrow{\frac{d\varphi_1}{dt}} + \overrightarrow{\frac{d\varphi_2}{dt}}\right) \times \frac{d}{dt}\vec{B}$$

Let's return now again to designations of differentiation by Newton (it is not for vectors since vectors and the designations appropriate to them were invented later [3, pp.350-352])

$$\ddot{\vec{B}} = \vec{\ddot{B}} + \left(\vec{\ddot{\varphi}}_1 + \vec{\ddot{\varphi}}_2\right) \times \vec{B} + \left(\vec{\dot{\varphi}}_1 + \vec{\dot{\varphi}}_2\right) \times \dot{\vec{B}}\,. \tag{7}$$

As sown in the formula (2) and Fig.1, angular speed of rotation of a vector of derivative length $\vec{\dot{B}}$ the same, as at the vector $\vec{B}$. It is visible in the same figure, that vector component $\vec{\dot{\varphi}}_1$ of the angular speed have constant direction along axis Y, that is its angular speed is equal to zero, and vector component $\vec{\dot\varphi}_2$ of the angular speed rotates with angular speed $\vec{\dot\varphi}_1$. Consequently, we can write:

$$\dot{\vec{B}} = \vec{\dot{B}} + \left(\vec{\dot\varphi}_1 + \vec{\dot\varphi}_2\right) \times \vec{B};$$

$$\dot{\vec{\dot\varphi}}_1 = \vec{\ddot\varphi}_1; \qquad \dot{\vec{\dot\varphi}}_2 = \vec{\ddot\varphi}_2 + \vec{\dot\varphi}_1 \times \vec{\dot\varphi}_2 .$$

Having substituted these expressions in (7), we shall receive

$$\ddot{\vec{B}} = \vec{\ddot{B}} + \left(\vec{\dot\varphi}_1 + \vec{\dot\varphi}_2\right) \times \vec{\dot{B}} + \left(\vec{\ddot\varphi}_1 + \vec{\ddot\varphi}_2 + \vec{\dot\varphi}_1 \times \vec{\dot\varphi}_2\right) \times \vec{B} + \left(\vec{\dot\varphi}_1 + \vec{\dot\varphi}_2\right) \times \left(\vec{\dot{B}} + \left(\vec{\dot\varphi}_1 + \vec{\dot\varphi}_2\right) \times \vec{B}\right)$$

Opening some brackets and grouping members with equal components of angular speeds, we shall receive finally

$$\ddot{\vec{B}} = \vec{\ddot{B}} + 2\left(\vec{\dot\varphi}_1 + \vec{\dot\varphi}_2\right) \times \vec{\dot{B}} + \left(\vec{\ddot\varphi}_1 + \vec{\ddot\varphi}_2\right) \times \vec{B} + \left(\vec{\dot\varphi}_1 \times \vec{\dot\varphi}_2\right) \times \vec{B} + \left(\vec{\dot\varphi}_1 + \vec{\dot\varphi}_2\right) \times \left(\left(\vec{\dot\varphi}_1 + \vec{\dot\varphi}_2\right) \times \vec{B}\right). \quad (8)$$

Members with double vector products it is possible to transform, then the formula of the second derivative of a vector will have some other kind

$$\ddot{\vec{B}} = \vec{\ddot{B}} + 2\left(\vec{\dot\varphi}_1 + \vec{\dot\varphi}_2\right) \times \vec{\dot{B}} + \left(\vec{\ddot\varphi}_1 + \vec{\ddot\varphi}_2\right) \times \vec{B} + \left(\vec{\dot\varphi}_1 + 2\vec{\dot\varphi}_2\right)\left(\vec{\dot\varphi}_1 \cdot \vec{B}\right) - \vec{B}\left(\dot\varphi_1^2 + \dot\varphi_2^2\right). \quad (9)$$

At use of the formula (4) for reception of the second derivative of a vector we shall not represent a derivative of a vector of angular speed $\vec\omega_B$ by same formula, that is hereinafter we shall not allocate a derivative from length of a vector of angular speed and a component because of change of a direction of this vector for simplification of a kind of formulas

$$\ddot{\vec{B}} = \dot{\vec{\dot{B}}} + \dot{\vec\omega}_B \times \vec{B} + \vec\omega_B \times \dot{\vec{B}} = \vec{\ddot{B}} + \vec\omega_B \times \vec{\dot{B}} + \dot{\vec\omega}_B \times \vec{B} + \vec\omega_B \times \left(\vec{\dot{B}} + \vec\omega_B \times \vec{B}\right) \equiv$$
$$\equiv \vec{\ddot{B}} + \vec\omega_B \times \left(\dot{\vec{B}} - \vec\omega_B \times \vec{B}\right) + \dot{\vec\omega}_B \times \vec{B} + \vec\omega_B \times \dot{\vec{B}} \quad (10)$$

Removing the brackets and uniting identical members in the first line of expression (10), we shall receive one more kind of the formula of the second derivative of a considered vector

$$\ddot{\vec{B}} = \vec{\ddot{B}} + 2\vec\omega_B \times \vec{\dot{B}} + \dot{\vec\omega}_B \times \vec{B} + \vec\omega_B \times \left(\vec\omega_B \times \vec{B}\right). \quad (11)$$

Let's open brackets and we shall unit identical members in the second line of expression (10). Then we shall receive one more kind of this formula where at the second member there is a vector of the first derivative of a vector instead of a vector of a derivative from its length

$$\ddot{\vec{B}} = \vec{\ddot{B}} + 2\vec\omega_B \times \dot{\vec{B}} + \dot{\vec\omega}_B \times \vec{B} - \vec\omega_B \times \left(\vec\omega_B \times \vec{B}\right). \quad (12)$$

It is analogously possible to transform and the formula (9), but we shall not make it. A member with double vector product in (11) and (12) it is possible to transform as two members which directions are clearer then the formula of the second derivative of a vector (12) will have one more some other kind

$$\ddot{\vec{B}} = \ddot{\vec{B}} + 2\vec{\omega}_B \times \dot{\vec{B}} + \dot{\vec{\omega}}_B \times \vec{B} - \vec{\omega}_B \left(\vec{\omega}_B \cdot \vec{B}\right) + \vec{B}\,\omega_B^2 . \tag{13}$$

All these kinds of the formula of the second derivative of a vector allow the same as and for the first derivative of a vector to give descriptive determination of the second derivative of a vector:

The second derivative of a vector is a vector being the sum of a vector of the second derivative length of a considered vector and vectors, received as a result of differentiation of a component of a vector of the first derivative which is resulting because of change of a direction of a considered vector concerning considered system of coordinates.

If to decompose a vector function in Taylor's series the increment of vector function can be presented as two groups, the first their which characterizes time variation of length of a considered vector, and the second group consists of the members which are turning out because of time variation of a direction of a considered vector concerning considered system of coordinates. Vectors of angular speed (or its components) of rotations of a vector and vectors of their derivatives enter therefore in members of the second group of members of an increment of vector function:

$$\vec{B}(t_0 + \Delta t) = \vec{B}(t_0) + \dot{\vec{B}}(t_0)\Delta t + \ddot{\vec{B}}(t_0)\Delta t^2 / 2 + \dddot{\vec{B}}(t_0)\Delta t^3 / 6 + \ldots =$$

$$= \vec{B}(t_0) + \left(\vec{B}(t_0) + \vec{B}(t_0)\Delta t / 2 + \vec{B}(t_0)\Delta t^2 / 6 + \ldots\right)\Delta t +$$

$$+\left(\vec{\omega}_B \times \vec{B}(t_0) + \left(2\vec{\omega}_B(t_0) \times \dot{\vec{B}}(t_0) + \dot{\vec{\omega}}_B(t_0) \times \vec{B}(t_0) - \vec{\omega}_B(t_0)\left(\vec{\omega}_B(t_0) \cdot \vec{B}(t_0)\right) + \vec{B}(t_0)\,\omega_B^2\right)\Delta t / 2 + \ldots\right)\Delta t$$

In kinematics an initial considered vector is the radius-vector $\vec{R}$ of a moving point, the first derivative from a radius-vector of a point is named in kinematics speed of this point and frequently designate $\vec{V}$, and the second derivative – acceleration (designate $\vec{W}$). After vectors $\vec{V}$ also $\vec{W}$ become known (in our case by formulas (1) and (9) or (4) and (13)), in some tasks there is a necessity of definition of components of acceleration $\vec{W}$ along a vector $\vec{V}$ and is perpendicular to it (so-called tangential acceleration and normal acceleration). As it was marked in [2, p.106], it by itself result at use of the formula (1) or (4) of the first derivative of a vector for definition of a vector of acceleration

$$\vec{W} \equiv \dot{\vec{V}} = \dot{\vec{V}} + \vec{\omega}_V \times \vec{V} \tag{14}$$

These components of a vector of acceleration should not be confused to two groups of components of a vector of the second derivative of a radius-vector $\vec{R}$ which can be received by formulas (8), (9), (11) – (13).

Let's emphasize once again, that if we examine a radius-vector of a moving point formulas (1) or (4) determine speed of a point of the end of this vector; if a considered differentiable vector is the vector of speed (the formula (14)) or any other physical vector the ends of these vectors do not correspond as the free vector can be transferred in parallel to any points, in particular, in the beginning considered CK, as shown in the Fig. 1.

4. Distinction of vectors of the second derivative of a vector at two systems of coordinates rotating from each other

In some works name invariant (not dependent on systems of coordinates) not only lengths of vectors, but also their directions [4, pp.35-36], confusing thus concepts "direction" and "orientation" (or meaning angles between two vectors). Orientation [3, pp.436-437] is an order (direction) of detour of any figure, more precisely, this instruction about an arrangement of points in one-dimensional space (this space can be and a nonplanar curve, broken line or the closed figure in three-dimensional space). Thus, these concepts coincide on direct lines as one-dimensional spaces (the instruction of a reference point, then positive and negative half-obstinate). But at transition to two-dimensional or three-dimensional spaces there is a necessity to specify position of a considered straight line concerning other straight lines, and it, according to F. Klejna's ideas; it is possible to make with the help of introduction of concept of a angle as metric measure. For example, the direction of movement is not only the indication of the positive half-tangent to a trajectory of movement of a point, but also position of this half-tangent concerning positive half-axle the chosen system of coordinates (it is possible to set a direction and with the help of two angles, as shown in the Fig.1, see also [2, pp.98-100]). The concept of a vector of speed has appeared after union of this concept with concept of scalar variable of speed as the time derivative of way [5, p.108].

So, directions we shall name angle only between vectors (or positive half-obstinate) and positive axes of coordinates or coordinate planes (see Fig.1) subject to algebraic sign (readout counter-clockwise). Orientation of vectors at any shifts and torsions is kept (that is vectors can not change the beginnings and the ends at absence of mirror reflections).

After it got discover that the direction of one considered vector concerning systems of coordinates with a various direction of axes is no invariant, that, angular speed of rotation of a vector concerning everyone SC also is no invariant (at rotation of systems of coordinates from each other), becomes clear, that the first derivative of a vector depends on everyone considered system of coordinates. It is fair for each subsequent derivative of a vector, hence, and for the second derivative of a vector.

Formulas of the second derivative of a considered vector $\vec{B}$ for two SC $OX_1Y_1Z_1$ and $OX_2Y_2Z_2$, rotating from each other with angular speeds $\vec{\omega}_2^{(1} = -\vec{\omega}_1^{(2}$, look like at use of the formula (12)

$$\ddot{\vec{B}}^{(1} = \ddot{\vec{B}} + 2\vec{\omega}_B^{(1} \times \dot{\vec{B}}^{(1} + \left(\dot{\vec{\omega}}_B^{(1}\right)^{(1} \times \vec{B} - \vec{\omega}_B^{(1} \times \left(\vec{\omega}_B^{(1} \times \vec{B}\right), \tag{15}$$

$$\ddot{\vec{B}}^{(2} = \ddot{\vec{B}} + 2\vec{\omega}_B^{(2} \times \dot{\vec{B}}^{(2} + \left(\dot{\vec{\omega}}_B^{(2}\right)^{(2} \times \vec{B} - \vec{\omega}_B^{(2} \times \left(\vec{\omega}_B^{(2} \times \vec{B}\right). \tag{15′}$$

To receive the formula of distinction or connection of these derivatives, it is necessary to subtract expression (15’) from (15) (it is possible and on the contrary). But before in (15) angular speed of a vector $\vec{B}$ and its derivative concerning the first SC we shall express in angular speed of a vector $\vec{B}$ and its derivative concerning the second SC

$$\vec{\omega}_B^{(1} = \vec{\omega}_B^{(2} + \vec{\omega}_2^{(1} \tag{16}$$

$$\left(\dot{\vec{\omega}}_B^{(1}\right)^{(1} = \left(\dot{\vec{\omega}}_B^{(1}\right)^{(2} + \vec{\omega}_2^{(1} \times \vec{\omega}_B^{(1} \text{ (by the formula (6))} \tag{17}$$

$$\left(\dot{\vec{\omega}}_B^{(1}\right)^{(2} = \left(\dot{\vec{\omega}}_B^{(2}\right)^{(2} + \left(\dot{\vec{\omega}}_2^{(1}\right)^{(2} \text{ (differentiating (16) concerning the second SC)} \tag{18}$$

Then the difference of the second members of expressions (15) and (15’) will look like

$$2\left(\vec{\omega}_B^{(2} + \vec{\omega}_2^{(1}\right) \times \left(\dot{\vec{B}}^{(2} + \vec{\omega}_2^{(1} \times \vec{B}\right) - 2\vec{\omega}_B^{(2} \times \dot{\vec{B}}^{(2} = 2\vec{\omega}_B^{(2} \times \dot{\vec{B}}^{(2} + 2\vec{\omega}_2^{(1} \times \dot{\vec{B}}^{(2} +$$

$$+2\left(\vec{\omega}_B^{(2} + \vec{\omega}_2^{(1}\right) \times \left(\vec{\omega}_2^{(1} \times \vec{B}\right) - 2\vec{\omega}_B^{(2} \times \dot{\vec{B}}^{(2} \quad = 2\vec{\omega}_2^{(1} \times \dot{\vec{B}}^{(2} + 2\left(\vec{\omega}_B^{(2} + \vec{\omega}_2^{(1}\right) \times \left(\vec{\omega}_2^{(1} \times \vec{B}\right)$$

The difference of the third members of expressions (15) and (15’) after substitution (16) and (18) in (17) looks like

$$\left(\left(\dot{\vec{\omega}}_B^{(2)}\right)^{(2} + \left(\dot{\vec{\omega}}_2^{(1}\right)^{(2} + \vec{\omega}_2^{(1} \times \left(\vec{\omega}_B^{(2} + \vec{\omega}_2^{(1}\right) - \left(\dot{\vec{\omega}}_B^{(2)}\right)^{(2)}\right) \times \vec{B} = \left(\dot{\vec{\omega}}_2^{(1}\right)^{(2} \times \vec{B} + \left(\vec{\omega}_2^{(1} \times \vec{\omega}_B^{(2}\right) \times \vec{B}$$

The difference of the fourth members of expressions (15) and (15’) will look like

$$\begin{aligned} &-\left(\vec{\omega}_B^{(2} + \vec{\omega}_2^{(1}\right) \times \left(\left(\vec{\omega}_B^{(2} + \vec{\omega}_2^{(1}\right) \times \vec{B}\right) + \vec{\omega}_B^{(2} \times \left(\vec{\omega}_B^{(2} \times \vec{B}\right) = \\ &= -\vec{\omega}_B^{(2} \times \left(\vec{\omega}_B^{(2} \times \vec{B}\right) - \vec{\omega}_2^{(1} \times \left(\vec{\omega}_B^{(2} \times \vec{B}\right) - \left(\vec{\omega}_B^{(2} + \vec{\omega}_2^{(1}\right) \times \left(\vec{\omega}_2^{(1} \times \vec{B}\right) + \\ &\quad + \vec{\omega}_B^{(2} \times \left(\vec{\omega}_B^{(2} \times \vec{B}\right) = -\vec{\omega}_2^{(1} \times \left(\vec{\omega}_B^{(2} \times \vec{B}\right) - \left(\vec{\omega}_B^{(2} + \vec{\omega}_2^{(1}\right) \times \left(\vec{\omega}_2^{(1} \times \vec{B}\right) \end{aligned}$$

The difference of the first members of expressions (15) and (15’) is equal to zero. Uniting other differences, we shall receive

$$\begin{aligned} \ddot{\vec{B}}^{(1} - \ddot{\vec{B}}^{(2} &= 2\vec{\omega}_2^{(1} \times \dot{\vec{B}}^{(2} + \left(\dot{\vec{\omega}}_2^{(1}\right)^{(2} \times \vec{B} + \vec{\omega}_2^{(1} \times \left(\vec{\omega}_2^{(1} \times \vec{B}\right) + \\ &+ \vec{\omega}_B^{(2} \times \left(\vec{\omega}_2^{(1} \times \vec{B}\right) + \vec{B} \times \left(\vec{\omega}_B^{(2} \times \vec{\omega}_2^{(1}\right) - \vec{\omega}_2^{(1} \times \left(\vec{\omega}_B^{(2} \times \vec{B}\right) \end{aligned} \qquad (19)$$

Based on properties of double vector product, the sum of last three members of expression (19) is equal to zero. Leaving in the left part of expression (19) first member, and in view of that the derivative of a vector of angular speed of rotation of two SC from each other does not depend on these SC (if is a little rotating SC derivatives of vectors of angular speeds will not depend only on those pairs SC which relative rotation they determine)

$$\left(\dot{\vec{\omega}}_2^{(1}\right)^{(2} = \left(\dot{\vec{\omega}}_2^{(1}\right)^{(1} - \vec{\omega}_2^{(1} \times \vec{\omega}_2^{(1} = \dot{\vec{\omega}}_2^{(1} \qquad (20)$$

finally we receive

$$\ddot{\vec{B}}^{(1} = \ddot{\vec{B}}^{(2} + 2\vec{\omega}_2^{(1} \times \dot{\vec{B}}^{(2} + \dot{\vec{\omega}}_2^{(1} \times \vec{B} + \vec{\omega}_2^{(1} \times \left(\vec{\omega}_2^{(1} \times \vec{B}\right) \qquad (21)$$

Clearly, that whereas the formula of the second derivative of a vector has some kinds also the formula of distinction of vectors of the second derivative in two systems of coordinates rotating from each other has some variants which can be useful in various theoretical or practical cases, but we shall not examine them here.

It is necessary to recognize, that above mentioned variant of a conclusion of the formula (21) not the shortest. Shorter conclusion turns out if to differentiate the formula of distinction or connection of the first derivatives of a considered vector at two systems of coordinates rotating from each other (6) concerning the first SC

$$\frac{d^{(1}}{dt}\left(\dot{\vec{B}}^{(1}\right) = \frac{d^{(1}}{dt}\left(\dot{\vec{B}}^{(2}\right) + \frac{d^{(1}}{dt}\left(\vec{\omega}_2^{(1}\right) \times \vec{B} + \vec{\omega}_2^{(1} \times \frac{d^{(1}}{dt}\left(\vec{B}\right) \qquad (22)$$

or the second SC

$$\frac{d^{(2}}{dt}\left(\dot{\vec{B}}^{(1}\right) = \frac{d^{(2}}{dt}\left(\dot{\vec{B}}^{(2}\right) + \frac{d^{(2}}{dt}\left(\vec{\omega}_2^{(1}\right) \times \vec{B} + \vec{\omega}_2^{(1} \times \frac{d^{(2}}{dt}\left(\vec{B}\right) \qquad (23)$$

Let's note that in formulas (22) and (23) there are cross derivatives, and we should compare the second derivatives at which the second differentiation is made concerning the same SC, as the first. Therefore further it is necessary in (22) in the right part for the first and third member of expression, and in (23) to a member of the right part of expression to apply the formula (6). With the account (20) these expressions at transition to designations by Newton will look like

$$\ddot{\vec{B}}^{(1} = \ddot{\vec{B}}^{(2} + \vec{\omega}_2^{(1} \times \dot{\vec{B}}^{(2} + \dot{\vec{\omega}}_2^{(1} \times \vec{B} + \vec{\omega}_2^{(1} \times \left(\dot{\vec{B}}^{(2} + \vec{\omega}_2^{(1} \times \vec{B}\right) \qquad (24)$$

$$\ddot{\vec{B}}^{(1} - \vec{\omega}_2^{(1} \times \left(\dot{\vec{B}}^{(2} + \vec{\omega}_2^{(1} \times \vec{B}\right) = \ddot{\vec{B}}^{(2} + \dot{\vec{\omega}}_2^{(1} \times \vec{B} + \vec{\omega}_2^{(1} \times \dot{\vec{B}}^{(2} \qquad (25)$$

Now it is visible, that at reduction of similar members of expression (24) and (25) turn in (21).

Usually make differentiation concerning the first SC, including its "absolute", that in the left part of expression to receive a "absolute" second derivative (the formula (22)). As we see, at our approach it has no any value as, naming considered SC the first and the second, we concerning any the third SC did not put forward any conditions of their movement. The deductive approach to construction of the theory helps to remove superfluous concepts, conditions and theorems.

Replacing a vector of angular speed $\vec{\omega}_2^{(1}$ a vector $-\vec{\omega}_1^{(2}$, we shall receive the formula, identical on structure with the formula (21), that is invariant:

$$\ddot{\vec{B}}^{(2} = \ddot{\vec{B}}^{(1} + 2\vec{\omega}_1^{(2} \times \dot{\vec{B}}^{(1} + \dot{\vec{\omega}}_1^{(2} \times \vec{B} + \vec{\omega}_1^{(2} \times \left(\vec{\omega}_1^{(2} \times \vec{B}\right) \qquad (26)$$

Let's note also, that is similar to the formula (6) for vectors of the first derivative of a vector of the formula (21) and (26) show, than vectors of the second derivative of a vector in two rotating from each other SC differ, but in them it is not visible, that in these derivatives identical. It becomes clear only at use of formulas (8)-(13) in the first variant of a conclusion of the formula (21), namely, that for a variable vector in all its derivatives (not dependent on systems of coordinates) all vectors its derivative lengths (module) are invariant. Though the first variant of a conclusion is longer, but from the scientific-methodical point of view it for this reason is better.

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Labendiks V. Klasiskas mehānikas kinemātikas deduktīvs izklāsts: vektora otrais atvasinājums vienā no divām koordinātu sistēmām, kuras rotē viena attiecībā pret otru

Saskaņā ar iepriekšējā rakstā autora piedāvāto pieeju (tajā pašā rakstu krājuma iepriekšējā pantā), ir aprakstīts deduktīva kinemātikas izklāsta variants uz Feliksa Klejna ģeometrisku ideju bāzes par telpas transformācijām, kustībām un diviem galvenajiem metriskiem lielumiem – attālumiem starp diviem punktiem (nogriežņa garumu) *un leņķi starp divām taisnēm un savu papildinājumu, proti, ieviešot savādāku mainīgā vektora un tā atvasinājuma definēšanu (ar attēlu palīdzību un formulām ar robežām). Autorā piedāvātās aprakstošās definīcijas (nosakāmā objekta īpašību apraksts, kas brīvā vektora gadījumā satur garumu un virzienu, bet kārtējiem atvasinājumiem – atbilstošos atvasinājumus pēc garuma un virziena). Detalizēti aplūkota vektora otrā atvasinājuma formulas izvešana sākumā vienā patvaļīgi izvēlētā koordinātu sistēmā, bet vēlāk divās sistēmās, kuras rotē viena attiecībā pret otru. Pievērsta īpaša uzmanība vektora otrā atvasinājuma jauno formulu veidojošām invariantām komponentēm, kurās izzūd šo atvasinājumu saiknes formula, kā arī parasti izmantojamajās paātrinājumu formulās gadījumā, kad koordinātu sistēmas rotē viena attiecība pret otru.*

Labendik V. Deductive enunciation of kinematics of the classical mechanics: the second derivative of the vector in one and two systems of coordinates rotating from each other
According to proposed in the previous article of the author (in the same scientific proceedings) of the way is proposed variant of a deductive enunciation of kinematics on the basis of Felix Klein's geometrical ideas about space transformations, motions and two basic metric variables ̃ distance between two points (segment length) and an angle between two straight lines, and also on the basis of offered by the author of the given article of replacement of constructive definitions of a variable vector and its derivatives (by means of drawings and formulas with limits) by descriptive definitions (exposition of properties of defined object, in case of the free vector it is the possession of a length and a direction, and for following derivatives – possession of corresponding derivatives of length and a direction). Such descriptive definitions become possible after a formula construction at which derivative lengths and directions are present in an explicit form. In detail is enunciated of the formula construction of a flexon of a vector at first in one any coordinate system, then in two coordinate systems rotating from each other. The attention is paid to invariant components of new formulas of a flexon of a vector in one any coordinate system, which disappear in formulas of connection of these derivatives (and also in usually used formulas of accelerations) in two coordinate systems rotating from each other.

Лабендик В. Дедуктивное изложение кинематики классической механики: вторая производная вектора в одной и двух вращающихся друг относительно друга системах координат
В соответствии с предложенным в предыдущей статье автора (в этом же сборнике статей) способом описан вариант дедуктивного изложения кинематики на основе геометрических идей Феликса Клейна о преобразованиях пространства, движениях и двух основных метрических переменных ̃ расстояние между двумя точками (длина отрезка) и угол между двумя прямыми, и также на основе предложенной автором данной статьи замены конструктивных определений переменного вектора и его производных (посредством рисунков и формул с пределами) дескриптивными определениями (описание свойств определяемого объекта, в случае свободного вектора – это обладание длиной и направлением, и для следующих производных – обладание соответствующими производными длины и направления). Такие дескриптивные определения становятся возможными после вывода формул, в которых производные длины и направления присутствуют в явном виде. Подробно рассмотрен вывод формул второй производной вектора сначала в одной произвольной системе координат, затем в двух вращающихся друг относительно друга системах координат. Обращено внимание на инвариантные составляющие новых формул второй производной вектора в одной произвольной системе координат, которые исчезают в формулах связи этих производных (а также и в обычно используемых формулах ускорений) в двух вращающихся друг относительно друга системах координат.