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Iterative Method for Solving a System of Linear Equations

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Abstract

The systems of linear equations are a classic section of numerical methods which was already known BC. It reached its highest peak around 1600-1700 due to the public demand for solutions of technical and engineering tasks, nevertheless, it is still topical nowadays. This paper describes another iterative approach to solving linear systems, which is based in the multiple transfers of the solution proximity point towards the solution itself, simultaneously reducing the differences of all the system equations.

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1. Introduction

A new demand for new means of solving systems of linear equations appeared at the same time as the computing technology emerged which promoted a rapid development of numerical methods for modelling physical processes by sampling (sub-dividing) the calculation range as well as replacing the differential operations by similar algebraic operations. According to the requirements of the final differences, final elements and their modifications, direct and iterative methods for approaching a poorly completed diagonal matrix with a strong main diagonal were developed¹.
². Methods for efficient storage of the equation system were developed, taking into account the symmetry of the

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matrix according to the main diagonal for both direct and iterative methods. In recent years, with the introduction of new numerical methods (super elements, the method of border elements), there has been a necessity for solving systems of linear equations with a completely filled matrix and one which does not possess the main diagonal dominance³. Iterative methods are often used for solving such tasks and the methods have been developed from the Gauss-Seidel method^{4,5}.

Solving systems of linear equations by iterative methods (such as Gauss-Seidel method) involves the correction of one searched-for unknown value in every step (see Fig. 1a) by reducing the difference of a single individual equation; moreover, other equations in this process are not used⁵. In order to accelerate the convergence of the iterative process, the methods are complemented by wellness principles which optimize the rate of the variable change in the iterative process.

The approach that is suggested in this article is based in every iteration simultaneously and is meant to reduce the differences of all the system equations by changing all the values of the unknown variables in the system (see Fig. 1b). The key question is how to organize the iterative process which is correcting the unknown variables and reducing the differences of all the equations at the same time.

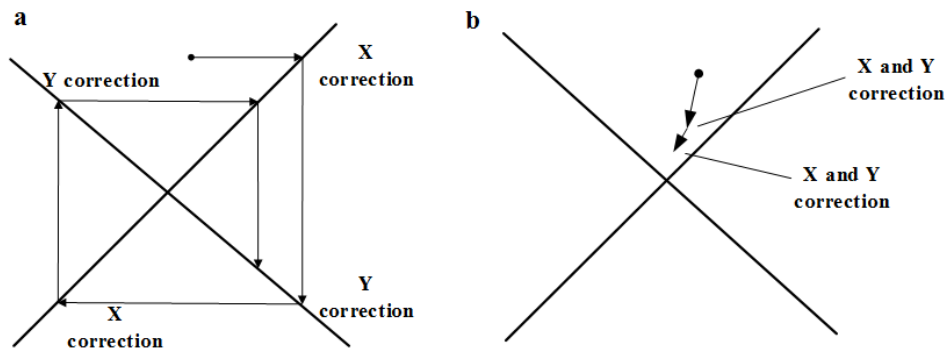


Fig 1. (a) the iterative solution according to the Gauss-Seidel method – a single variable is corrected in every step; (b) the provided iterative solution scheme – all the variables are corrected in every step.

Initially, the solution for the two equations system is to be viewed, and then it is to be generalized to any number of final dimensions.

2. The essence of the method for a two equations system

The initial solution approximation $P^0(x, y)$ is freely assumed. It can easily be shown that any point $P(x, y)$ in the XY plane is located further from the solution of the equation system than the projection $P(x, y)$ of equations (see Fig. 2). Subsequently, in order to solve the equation systems the following iterative process can be implemented in Fig. 2:

- The proximity $P^i(x, y)$ projection on the equations are found
- The midpoint (or arithmetic mean) is assumed as the new proximity $P^{i+1}(x^{i+1}, y^{i+1})$ between the projections of the point $P^i(x^i, y^i)$ on the system equations a and b
- The correction of the proximity is iteratively repeated

In the case of solving iterative linear systems, point $P(x, y)$ is moved step by step – $P^0(x^0, y^0)$, $P^1(x^1, y^1)$, $P^2(x^2, y^2)$, ... – until a certain state has been found in which the proximity of the solution is close enough to the solution $R(x, y)$ of the systems of linear equations. Since the solution $R(x, y)$ is sought for, the quality of the proximity has to be assessed, for instance, by applying the approximate solution in the equations and evaluating the differences, determining that the proximity no longer changes in the iterative process etc.

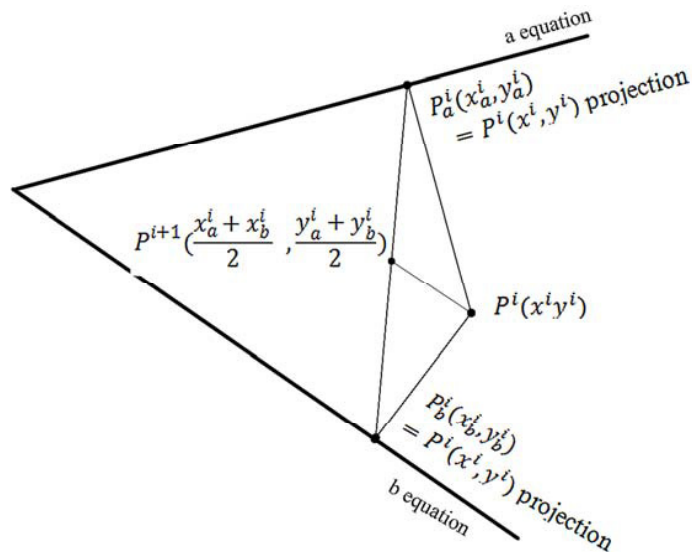


Fig. 2. Location of the current proximity in the iterative process.

It is easy to prove that:

- Given a normal system, the proposed iterative process approaches the solution in every step of the way
- As a result of the iterations the proximity of the solution will not change the range in which the system resolution takes place
- The rate of the convergence of the iterations process depends on the characteristics of the equation system or in the case of two dimensions it is the angle which consists of equations (see Fig. 3)
- if the equations are contradictory (the lines are close to the parallels), the process will not converge and there will be no solution obtained
- Contrary to the popular methods, this approach has no strict requirements for the matrix of the equation system – the dominance of the main diagonal is not significant
- The increase of the proximity of the solution, whose direction is close to optimal, but the length is usually significantly less than optimal, is determined in each iteration and it is necessary to develop methods for correcting the increase of the proximity of the solution

3. The approach in case of larger systems

The reviewed approach is easily applicable to any final number of equations. In the case of two dimensions, the geometric equation is a line, but a non-contradictory equation system has two lines that intersect, thus forming 4 angles. The process of the iterative solution takes place in only one angle.

In the case of three dimensions, one equation is a geometric plane; a non-contradictory equation system is three planes all of which intersect with one another. The process of the iterative solution takes place inside the pyramid whose surface is composed by three plane equations and the apex of a pyramid is searched for. There are 8 pyramids where in only one of the pyramids the solution is sought for.

In the case of more dimensions, the equation is an n -dimensional hyperplane where n is the layer of the equations system. A non-contradictory system of equations is the set of n -hyperplanes that all mutually intersect. The process of the iterative solution takes place inside the hyperpyramid whose hyper surface consists of n -hyperplane equations and the apex of the pyramid is searched for. In the case of four and more dimensions it is impossible to imagine the case subject graphically, but all the mathematical approaches and formulas still remain.

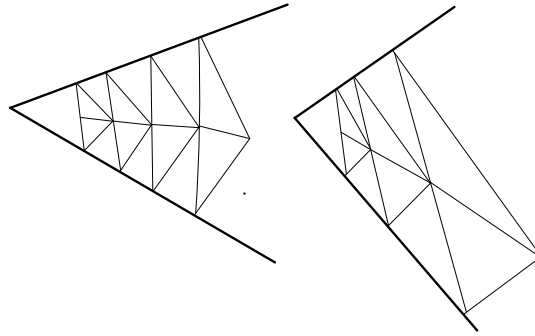


Fig. 3. The dependence of the iterative process convergence rate (iteration step) of the characteristics of the equation system.

In the case of any number of dimensions, the projection coordinates of the point $P(p_1, p_2, \dots, p_n)$ on the hyperplane:

$$a_1 * x_1 + a_2 * x_2 + \dots + a_n * x_n + b = 0 \quad (1)$$

can be calculated⁵ as follows.

The normal vector $n = (a_1, a_2, \dots, a_n)$ of the plane (1) is perpendicular to the plane; therefore, it is also the direction vector of the plane which is the shortest distance from the point $P(p_1, p_2, \dots, p_n)$ to the plane, i.e., to the projection of the point in the plane.

If the canonical equations of the line are completed with the following coordinates of the point $P(p_1, p_2, \dots, p_n)$:

$$(x_1 - p_1)/a_1 = (x_2 - p_2)/a_2 = \dots = (x_n - p_n)/a_n \quad (2)$$

- The linear equation (2) is expressed in parametric way:

$$\begin{aligned} (x_1 - p_1)/a_1 &= t, \text{ or } x_1 = a_1 * t + p_1 \\ (x_2 - p_2)/a_2 &= t, \text{ or } x_2 = a_2 * t + p_2 \\ (x_n - p_n)/a_n &= t, \text{ or } x_n = a_n * t + p_n \end{aligned} \quad (3)$$

- By inserting x_1, x_2, \dots, x_n in the equation of the plane, it results in:

$$t = - \frac{a_1 * x_1 + a_2 * x_2 + \dots + a_n * x_n + b}{a_1 * a_1 + a_2 * a_2 + \dots + a_n * a_n} \quad (4)$$

- The coordinates of the projection point $P(p_1, p_2, \dots, p_n)$ will be:

$$\begin{aligned} P_{1proj} &= a_1 * t + p_1 \\ P_{2proj} &= a_2 * t + p_2 \\ &\dots \\ P_{nproj} &= a_{1n} * t + p_{1n} \end{aligned} \quad (5)$$

In the case of any number of dimensions, the new value of the proximity coordinates is taken as the arithmetic mean of the previous projection of the proximity on hyperplanes.

4. Optimal selection of the increase of proximity in the iterative process

The change of the proximity of the equation system solution slowly converges (see Fig. 4) by the means of the approach described above, since in a single iteration step the increase in the calculated proximity coordinates is

lower than optimal (maximum permissible).

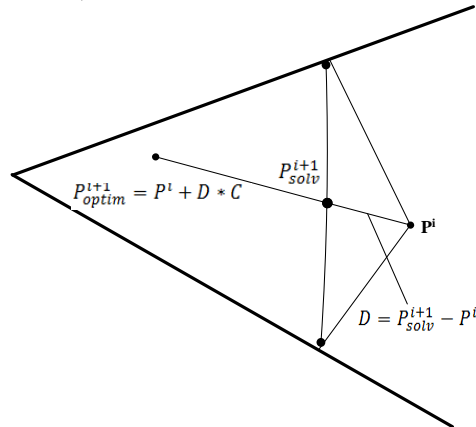


Fig. 4. The optimal choice of the current proximity P^{i+1}_{solv} , the $D = P^{i+1}_{solv} - P^i$ estimated increase, C - optimizing factor.

The determination of the optimal increase of the solution proximity, while maintaining the calculated direction, is a constant problem. The determination of the optimal value of the factor C requires excessive calculations and it is necessary to calculate the differences of all the equations in every step of the way. A method for determining the optimal value of C without the need for excessive calculations has not been developed yet. The factor C can be determined empirically by gradually increasing the value of C and calculating the differences of the equations; if any of the equation differences changes its sign, it means the factor C already exceeds its permissible value. Furthermore, it is empirically determined that a certain optimal C value can be applied in a number of iterations.

5. Conclusion

Another iterative approach for solving linear systems of equations has been offered and it is based on simultaneous minimization of all invalid links of equations. The given approach mainly does not impose any requirements of the equation system. The approbation of the methods has been done which proves that the methods are perfectly valid; however, the distribution of the method in each iteration requires considerable calculations. The reduction of the amount of calculations is a matter of a separate study. Interesting results have been achieved, but there needs to be further work on it in order to develop and improve the approach.

References

1. Demidovich B, Maron I. The basics of numerical methods. Moscow: Nauka; 1970. (in Russian).
2. Bronshtein I, Semendjajev K. Mathematical manual for engineers and students. Moscow: Nauka; 1981. (in Russian).
3. Tsybenko A, Konyukhov A, Tsybenko H. Numerical Method for Determining Stiffness Characteristics of an Arbitrary Form Superelement. *Applied Computer Systems*. 18; 2015. p. 52-56.
4. Heck A. Introduction to MAPLE. Springer-Verlag; 1996.
5. Trott M. The Mathematica guidebook for numerics. Springer Science; 2006. 1208.
6. Available: <https://estudijas.rtu.lv/course/view.php?id=38111>.



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